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TOPICS IN ALGEBRAIC GEOMETRY

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0.1. **Grassmannians.** Study the family of all linear subspaces of a vector space. Especially recommended if you are interested in representations of groups like GL_n etc.

Ref. [Has07, Chap.11]

Extensions: [LB09].

0.2. **Riemann–Roch on curves.** Give a formula for the dimension of the space of functions with preassigned poles. Much of algebraic geometry starts here.

Ref. [Ful89, Chap.8]

Extensions: You should really learn some cohomology if you plan to go further. The standard text is [Har77, Chap.III]. Good luck!

0.3. **Groebner bases.** As you will see, algebraic geometry is good with existence theorems. Groebner bases give a method that allows one to do actual computations.

Ref: [CLO97, Chap.2]

Extensions: learn the program *Macaulay* and do some work with it. (With this you will be mostly on your own since I never used it.)

0.4. **Examples of algebraic group actions.** Through examples, study algebraic groups like GL_n, PGL_n and their actions.

Ref. [Har95, Lect.10]

Extensions: This is a vast topic, one can start with [Hum75] or [Bor91].

0.5. **Curve singularities.** Local study of plane curves, that is, given a plane curve $C := (f(x,y) = 0)$, we try to understand C in a small neighborhood of the origin. First step: near the origin we try to write y as a function of x and give some kind of infinite series expansion.

Applications: resolution of singularities and understanding the topology of $C \cap (|x|^2 + |y|^2 = \epsilon^2)$ as a subset of the 3-sphere $(|x|^2 + |y|^2 = \epsilon^2)$ of radius ϵ.

Ref: [BK86, Sec.8]

Extensions: This is long enough, but one can do other resolution methods [Kol07b, Chap.1] or higher dimensional singularities [AGZV85].

0.6. **Rational varieties.** Mostly through examples like plane conics, quadrics and cubics in $\mathbb{P}^3$ study the geometry and arithmetic of rational varieties.

Ref: [KSC04, Chap.1]

Extensions: [Kol08, Secs.1–3] or [Kol02] or [KSC04, Chap.2].

0.7. **Elliptic curves.** Essentially the study of plane cubic curves and their other incarnations. The geometry is well understood; many deep open number theoretic questions remain.

Refs. The most elementary is [Rei88, §2]. You should also go through [Cas91, Secs.6–9].

Extensions: [Sil09] and much that lies beyond.

0.8. **Elliptic functions.** A beautiful treatment is in [Sie88, Chap.1]. Needs only the basics of 1-variable complex analytic functions.

Extensions. You can continue with [Sie88].

0.9. **Hasse principle.** First prove that a quadric over $\mathbb{Q}$ has a point iff it has a point over $\mathbb{Q}_p$ for every p. Then show that the analogous statement fails for cubic surfaces.

Ref. [Cas91, Secs.1–5], [Ser73, Chap.IV] and [SD62, Mor65].

Extensions: (I am still looking.)

0.10. **Tarski-Seidenberg Theorem.** A subset $X \subset \mathbb{R}^n$ is basic semialgebraic if it is given by conditions $p_i(x_1, \dots, x_n) \geq 0$ where the p_i are polynomials. Taking finite unions, intersections, complements we get semialgebraic sets.

Theorem. The projection of a semialgebraic set is again a semialgebraic set.

Start with the complex case: Chevalley's theorem that images of algebraic varieties are constructible sets.

Ref: [BCR98, Chap.2]

Possible extension: What should be the right notion for subsets of $\mathbb{Q}_p^n$?

0.11. **Chevalley's theorem on invariants of finite groups.** Let $G \subset GL(n,k)$ be a finite group. We get an action on $k[x_1, \dots, x_n]$. The question is: what is the subring of invariants $k[x_1, \dots, x_n]^G \subset k[x_1, \dots, x_n]$.

Theorem. If $k = \mathbb{R}$ and G is generated by reflections then $k[x_1, \dots, x_n]^G$ is again a polynomial ring.

Ref: [Che55]

Extensions: You should work out the corresponding result for $k = \mathbb{C}$ and other fields. Other directions: [Ben93, Chaps.1–3].

0.12. **Simple singularities.** We work with power series $f(x_1, \dots, x_n)$. Two power series f, g are considered equivalent if there is a coordinate change given by power series $x_i \mapsto \phi_i(\mathbf{x})$ such that $f(\phi_1(\mathbf{x}), \dots, \phi_n(\mathbf{x})) = g(\mathbf{x})$.

Given two power series f, g, we can view $f + \epsilon g$ as perturbations of f. A very fruitful question of singularity theory asks: what can we say about the perturbations of a polynomial or power series f?

The aim is to classify those power series $f(x_1, \dots, x_n)$ that have only finitely many inequivalent perturbations.

Ref. Probably the best is to think about this and then get the proof from [KM98, 4.24–25] by replacing Steps 6 and 9. Or you can look at the general case in [AGZV85, Secs.11–].

Extensions: More than you want is in [AGZV85, Secs.11–15].

0.13. **Chow's theorem.** This is the following.

Theorem. Let $Z \subset \mathbb{CP}^n$ be a Euclidean closed subset that is locally definable as a common zero set of analytic functions. Then Z is algebraic, that is, globally the common zero set of polynomials.

It is helpful if you are somewhat familiar with several variable complex analytic functions.

Ref: [Mum95, Chap.4].

Extensions. If you are up to it, read [Ser56].

0.14. **Minimal degree varieties.** The aim is to classify irreducible subvarieties of $\mathbb{P}^n$ that are not contained in any linear subspace and whose degree is as small as possible. Nice concrete geometry.

Ref: [EH87]

Extensions: minimal multiplicity local rings [Sal79]; Castelnuovo bound for space curves [ACGH85, Sec.III.2]; other extremal examples [Rus00, CMR04].

0.15. **Pointless varieties and large fields.** The aim is to show that if k is a field such that there is a geometrically irreducible k-variety without k-points (for instance if $k = \mathbb{R}$ then the conic $(x^2 + y^2 + z^2 = 0) \subset \mathbb{P}^2$ is such) then there is also such a plane projective curve.

Ref. [Kol07a, Sec.1]

Extensions. Best is to study the Weil estimates for points over finite fields (I am still looking for an elementary introduction.).

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Keywords: Grassmannians; Karcher Mean; Face Recognition; Illumination Spaces; Compressions; Feature Patches; Principal Angles. 1. Introduction.Â The linear span of the columns of this matrix forms a k-dimensional vector subspace in Rn , which can be realized naturally as a. point on the Grassmannian GR. (k, n.) . The detail of this.Â We will, however, group k (generally in-dependent) example images of a subject and consider the k-dimensional feature subspace they span in Rn . The connection between this linear subspace to a point on the Grassmann manifold will be made precise next. Definition 2.1 The Grassmannian GR. (k, n.) or the. Grassmann manifold is the set of k-dimensional suspaces in an n-dimensional vector space Kn for some field K. We investigate subspaces when they intersect trivially. In this case, the union of bases for these subspaces are basis for the entire vector space. Direct sum.Â Vector Space of Polynomials and Coordinate Vectors Let P_2 be the vector space of all polynomials of degree two or less. Consider the subset in P_2 \[Q=\{ p_1(x), p_2(x), p_3(x), p_4(x)\},\] where \begin{align*} &p_1(x)=x^2+2x+1, &p_2(x)=2x^2+3x+1, \\ &p_3(x)=2x^2, &p_4(. x)=2x^2+x+1. \end{align*} (a) Use the basis [â€¦]Â Express a Vector as a Linear Combination of Other Vectors. Prove that $\{ 1 , 1 + x , (1 + x)^2 \}$ is a Basis for the Vector Space of Polynomials of Degree 2 or Less. How to Find a Basis for the Nullspace, Row Space, and Range of a Matrix. Study Guides. Linear Algebra. Projection onto a Subspace. All Subjects. Vector Algebra. The SpaceR2.Â Example 2: Let S be a subspace of a Euclidean vector space V. The collection of all vectors in V that are orthogonal to every vector in S is called the orthogonal complement of S: (S âŠ¥ is read â€œS perp.â€) Show that S âŠ¥ is also a subspace of V. Proof. First, note that S âŠ¥ is nonempty, since 0 âˆˆ S âŠ¥. In order to prove that S âŠ¥ is a subspace, closure under vector addition and scalar multiplication must be established. Let v 1 and v 2 be vectors in S âŠ¥; since v 1 Â· s = v 2 Â· s = 0 for every vector s in S, proving that v 1 + v 2 âˆˆ S âŠ¥. Therefore, S âŠ¥ is closed under vector addition. Question 1: What is the name for images that are simplified, pictorial representations of actual things, like the two images shown above? Symbols. Icons.Â Despite the modernization of production techniques, a lot of vernacular design still looked a lot like the handcrafted objects of the past. In the last two decades of the 19th century, the production of wooden household goods surpassed metal and ceramic goods combined.Â Graphic Design. Textiles. Material Studies. Question 4: What was a key characteristic of some of the crafted objects, such as ceramics, produced by students of the Bauhaus? They were objects intended for mass manufacture but had the same level of craft that traditionally went into one-of-a-kind objects applied to them.