Radical parametrizations of algebraic curves

Wurzelparametrisierungen von algebraischen Kurven

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Abstract

It is well known that an algebraic curve can be parametrized by rational functions if and only if its genus is zero. The aim of this project is to study a more general class of curves: those parametrizable by radicals, that is, which admit a parametrization involving field operations and root extractions. This class is significantly larger than the previous one: it contains all elliptic and hyperelliptic functions, thus there are curves of every genus which have radical parametrizations.

We approach the problem from an algorithmical point of view. In other words, our goal is to devise the best possible algorithms that decide whether a curve can be parametrized by radicals and compute one/all/the best radical parametrizations in the affirmative case. Naturally, this involves theoretical research into structural problems like a good definition of what is a better parametrization, what can we say about the structure of the class of all radical parametrization of a curve, etc. These questions go back to Zariski.

Zusammenfassung

Eine algebraische Kurve besitzt bekanntlich dann und nur dann eine rationale Parametrisierung, wenn ihr Geschlecht gleich Null ist. Ziel dieses Projekts ist es, eine allgemeinere Klasse von Kurven zu analysieren: jene, die eine Parametrisierung durch Radikale besitzen, d. h. eine Parametrisierung bestehend aus Körperoperationen und Wurzeln. Diese Klasse ist bedeutend größer als die obige: Sie enthält alle elliptischen und hyperelliptischen Kurven, es gibt also Kurven beliebigen Geschlechts, welche eine Wurzelparametrisierung zulassen.

Unser Zugang zu dieser Fragestellung ist ein algorithmischer Standpunkt. Mit anderen Worten, unser Ziel ist die Entwicklung der bestmöglichen Algorithmen zur Entscheidung, ob eine Kurve durch Radikale parametrisiert werden kann, und zur Berechnung von einer/aller/der bestmöglichen solchen Parametrisierung. Natürlich setzt dieses Ziel theoretische Forschungen über strukturelle Fragen voraus, wie etwa die Frage nach einer angemessenen Definition einer "besseren Parametrisierung", oder die Frage nach der Beschaffenheit der Klasse aller Wurzelparametrisierungen einer Kurve, etc. Diese Fragen gehen zurück auf Zariski.

Contents

1 Scientific aspects of the project

1.1 Background

Algebraic varieties (that is, subsets of some n-dimensional space defined as the zeroset of polynomials) are a natural object of study due to its combination of rich structure and rigidity. Among them, algebraic curves (1-dimensional sets) are the simplest nontrivial objects, yet there are a lot of questions that are simple to state but difficult to answer to a high degree of satisfaction. One of these questions is the object of our proposal.

Curve parametrization is a classical topic in algebraic geometry and it has applications to, among other fields, Computer Aided Design, as demonstrated by the numerous papers on rational parametrization in the journal *Computer Aided Geometric Design*. In this discipline, curves arise sometimes in implicit form, for instance as intersections of two surfaces in space or projections of such intersections, or contour curves. For vector graphics (magnifying, moving, etc.) it is very convenient to have a parametric representation.

In a broad sense, given a curve defined as the vanishing locus of finitely many polynomials $F_1(x_1,\ldots,x_n),\ldots,F_k(x_1,\ldots,x_n)$, a parametrization is a n-tuple of functions $\gamma_1(t),\ldots,\gamma_n(t)$ in one variable t such that $F_1(\gamma_1(t),\ldots,\gamma_n(t)) = \cdots = F_k(\gamma_1(t),\ldots,\gamma_n(t)) = 0$. Since the image of a parametrization is irreducible, we restrict to the discussion to irreducible curves. Also, since every curve is birationally equivalent to a regular curve, all curves will be regular from now on.

The functions $\gamma_i(t)$ can belong to different classes of functions, resulting in different types of parametrizations. Clasically, they are rational functions in t. In that case, a curve that admits a parametrization is called *rational*. It is well-known that rational curves are exactly the curves of genus zero. Many other results on rational parametrizations are known, and in addition efficient algorithms exists to decide whether a curve is rational (in fact to calculate its genus), to calculate a parametrization in the affirmative case, to calculate the simplest parametrization according to several different criteria (properness, coefficient field, etc.) See [SWPD08] for many more details. Unfortunately, some natural constructions, like the offset of a curve, do not preserve rationality, see [ASS97].

A larger, natural class of parameter functions that one can consider is those which involve the four field operations, plus extraction of roots of any index. We call *radical* a parametrization by such functions. This extension greatly enlarges the class of parametrizable curves. For example, there are curves of every positive genus which are radically parametrizable, namely all elliptic and hyperelliptic curves. Those are birationally equivalent to plane curves with equations of the form

$$y^2 = P(x), \quad P(x) \text{ polynomial of degree} \geq 3$$

which immediately lead to the parametrization $x = t, y = \sqrt{P(t)}$. This generalizes easily to curves with equation $F(x,y) = 0$ where the degree of F in one of the variables is less or equal than 4, since we can then solve that variable in terms of the other by radicals. In fact, it is proved in [Zar26] that every curve of genus less or equal than 6 admits a $4:1$ map to the projective line, thus being radically parametrizable.

The above suggests the relevance of Galois theory in this problem. We elaborate in the next section.

1.1.1 Galois theory for radical parameterizations

A classical theorem in Galois theory is the following.

Theorem 1. Let K be a field of characteristic 0. A polynomial in $K[x]$ is solvable if and only if its Galois group is solvable.

If we apply this to the equation of a plane curve $F(x, y) = 0$ seen as a polynomial in y, we obtain that y is expressible by radicals in terms of x if and only if the Galois group of $F \in K(x)[y]$ is solvable. This corresponds to the solvability of the projection map $(x, y) \mapsto x$.

This situation can be generalized in a straightforward way. Given a curve C and a map $\phi \colon C \to \mathbb{P}^1$, the (multivalued) inverse of ϕ can be expressed by radicals if and only if the Galois group of the splitting field of the extension $K(C)/K(\mathbb{P}^1)$ is solvable. This group is called the *monodromy group* of the curve, and it can be seen precisely as the monodromy of a branched covering map between Riemann surfaces.

1.1.2 Properties of rational parametrizations

The following are some facts about rational curve parametrizations whose radical analogues are worth studying.

- Over the complex numbers, every parametrization of a regular curve is surjective onto $\mathbb{P}^1$.
- Every parametrization can be written as the composition of a *proper parametrization* (injective except at finitely many points) with a rational function $R(t)$. Given a parametrization, one can compute efficiently a proper parametrization of the curve.
- Let K be the field generated by the coefficients of a rational curve C. Then C admits a rational parametrization over a finite extension of K. An *optimal parametrization* is one whose coefficients generate the smallest possible extension of K (this may be strictly larger than K). The existence of a rational parametrization with coefficients in $E \supset K$ is equivalent to the existence of a regular point in C with coefficients in E. Thus finding an optimal parametrization corresponds to finding regular points on the curve whose coordinates generate the smallest possible extension of K. Finally, a curve of odd degree over K has an optimal parametrization over K, and a curve of even degree has an optimal parametrization over K or a degree 2 extension of K, depending on whether a certain conic has regular points over K or not. See Section 5.1 of [SWPD08] for results and Section 5.3 for an efficient algorithm.

Of these, the issue of proper parametrizations is known to be significantly more subtle for radical parametrizations that it is for rational parametrizations: in [PS05] the authors exhibit an algebraic curve X that does not admit a solvable map $X \to \mathbb{P}^1$ but for which there exists a finite covering $X' \to X$ where X' admits a solvable map to $\mathbb{P}^1$.

Also, the field of definition poses complications, inasmuch as roots may be undefined for infinitely many points, or at all.

1.2 State of the art

The literature on the problem of radical parametrizations of curves is scarce. Possibly the first published result in this area is [Zar26], where a question by Enriques in the Congress of Mathematicians of 1897 in Zurich is solved: prove that, given a general algebraic equation in two variables $f(x, y) = 0$, it is not possible to introduce a parameter t, rational function of x and y, such that x, y can be written as a function of t by radicals. Zariski proves that for genus $g > 6$, a generic curve cannot be parametrized in this way.

He also poses the following question, expressed here in modern terms: is it true that, for a general curve X, there exists no finite covering $X' \to X$ where X' can be parametrized by radicals? In [PS05] the authors exhibit a curve X that does not admit a solvable map $X \to \mathbb{P}^1$ but for which there exists a finite covering $X' \to X$ where X' does admit a solvable map to $\mathbb{P}^1$.

In relation to this, [Sen08] provides algorithms to parametrize by radicals any curve of genus $g \leq 4$.

1.3 Goals

The main goal of this project is to produce an algorithm that, given an algebraic curve C, outputs a rational map $C \to \mathbb{P}^1$ which is invertible by radicals (or the parametrization), or "none" if no radical parametrization exists. For several lines of research in this direction, see the next section.

In principle we will work over the field of complex numbers. This makes many theoretical problems much simpler and allows us to have the best machinery of algebraic geometry at our disposal. From a computational point of view, we will be in reality working over the algebraic closure of the field of rational numbers; all relevant computations can be performed in an exact way (a classical reference on computational number theory is [Coh93]). Other interesting cases, like the real numbers, may be considered too.

1.4 Methods

1.4.1 Monodromy and Galois theory

To explore monodromy and Galois theory in order to connect the algorithms in those areas with the problem of radical parametrizations, see Section 1.1.1. In particular, it seems that the ways of projecting C down to $\mathbb{P}^1$ are relevant objects of study, see also Section 1.4.3. It is important to note that this is related to inverse Galois theory [MM99, Völ96] but our problem is of a different nature: we do not have a field extension to analyze, rather we have the field of an algebraic curve but a suitable subfield of it is not given.

1.4.2 The case of degree 4 covers

Definition 1. A curve is said to have *gonality* n if it admits a $n : 1$ map to $\mathbb{P}^1$ but no map of smaller degree.

The curves of gonality two are called *hyperelliptic*. As mentioned before, all hyperelliptic curves have radical parametrizations, since they can be written as

$$y^2 = P(x), \quad P(x) \text{ polynomial of degree} \geq 4.$$

It is clear that curves with gonality 3 and 4 are also parametrizable by radicals. Curves of gonality 3 (also called *trigonal*) arise in a natural way in the study of canonical curves:

Theorem 2. Let C be the image of a non-hyperelliptic curve by its canonical map. Then, either

1. C is entirely cut out by quadric hypersurfaces; or
2. C is trigonal, in which case the intersection of all quadrics containing C is the rational normal scroll swept out by the trichords of C; or
3. C is a plane quintic, in which case the intersection of the quadrics containing C is the Veronese surface in $\mathbb{P}^5$, swept out by the conic curves through five coplanar points of C.

Proof. See [Enr19] and [Bab39] for the original proof, [SD73] for a modern treatment, also [GH78, p. 535]. □

We have recently developed an algorithm that decides whether a given curve is trigonal and produces a $3 : 1$ map to $\mathbb{P}^1$ in the affirmative case, see [SS09]. We make use of the Lie algebra method explained in [dGPS09] in order to decide the type of the surface and obtain the desired map.

The remaining case, namely *tetragonal* curves, is another of our goals. In this case, rather than trichords cutting out the $3 : 1$ map, it is a pencil of surfaces which cuts out the quadruple of points of each fiber. We currently ignore what the variety swept by those planes is; we know it to be a determinantal variety. We intend to study it in order to determine if the Lie algebra method can be applied, and to search for other constructive methods in the negative case.

1.4.3 Quotients of curves

We assume that all our curves are defined over a field of constants K. The following property holds: for every non-constant map $\phi\colon C_1 \to C_2$ between algebraic curves, we have that $g(C_1) \geq (C_2)$. One possible way of constructing a map from a given curve C to a lower genus curve is to consider projections from C to $C' = C/\langle\sigma\rangle$ where σ is an automorphism of C. This results in a field extension $K(C')/K(C)$ whose Galois group is cyclic. If by repeating this procedure we reach one of the possible base cases (genus 0, genus 1 or hyperelliptic, for example) then we can compose the resulting maps to obtain a radical parametrization of the original curve. The problem with this approach is that it is possible that a curve is parametrizable by radicals but a solvable map cannot be written as a composition of cyclic covers. Therefore it is interesting to know to which extend this algorithm will provide a parametrization if one exists.

Additionally, the computational aspect of quotients of curves by automorphisms is not trivial. We have taken some steps in the direction of a new method to compute the automorphism group of an algebraic curve since the currently existing implementations are too slow. Our approach is the following: a curve has finitely many Weierstrass points, which are permuted by all automorphisms. The method that we intend to explore, due to [Hes04], can be roughly described in the following way:

1. Calculate the Weierstrass points of the curve.
2. Choose a Weierstrass point P.
3. For each Weierstrass point, calculate all the automorphisms of the curve which send P to it.

The Weierstrass points are exactly the points where the Wronskian determinant vanishes (see [Hur92]). In order to calculate the automorphisms of the curve that send one point to another, we can do this in the canonical model (we assume non-hyperellipticity) so that all automorphisms are in fact linear automorphisms of $\mathbb{P}^{g-1}$. On the other hand, we intend to improve this algorithm by producing another way of computing these automorphisms, avoiding the canonical model.

Our approach consists on considering the Riemann-Roch spaces of the two points P and Q, since $\mathcal{L}(nP)$ must be isomorphic to $\mathcal{L}(nQ)$ for every n. We have introduced what we call the *Tschirnhaus-Weierstrass form* of a curve, see [SS08], which has the following desirable property.

Theorem 3. Let C_1, C_2 be curves in Tschirnhaus-Weierstrass form with the same normal forms and good monomials [this is some combinatorial data, fixed in advance]. If $\varphi\colon C_1 \to C_2$ is an isomorphism, it is given by a diagonal matrix, i.e. $\varphi(x_1, \ldots, x_r) = (y_1, \ldots, y_r)$ with $y_i = \lambda_i x_i$.

It is necessary to explore the properties of this algorithm, implement it, and of course to continue the research in the bigger problem from which this "excursion" arose.

1.4.4 Abelian varieties

Every curve has an algebraic variety associated to it which is called its *Jacobian*. The Jacobian of a curve of genus g is a variety of dimension g and it also has a group structure compatible with it (meaning that the product and inverse maps are regular). Varieties of this type are called *abelian varieties*. The concept is related (but not equivalent) to that of *complex tori*, that is, quotients of K^g by lattices $\Lambda \subset K^g$ of rank $2g$; every abelian variety is a complex torus, but not conversely. However, every complex torus of dimension 1 is an abelian variety.

A classical theorem on abelian varieties is Poincaré's Reducibility theorem ([Poi86] or [BL04]), which states that every abelian variety can be written as a product of *simple* abelian varieties up to isogeny.

Definition 2. An abelian variety is *simple* if it does not contain a proper abelian subvariety.

Definition 3. Two abelian varieties A, B are *isogenous* if there exists a surjective morphism $A \to B$ whose kernel is finite. Being isogenous is an equivalence relation.

Theorem 4. Every abelian variety A is isogenous to a product of simple abelian varieties.

In our context, the Jacobian $J(C)$ of a curve C can be decomposed as a product $A_1 \times \cdots \times A_k$ of simple abelian varieties. If one of these has dimension one, we have a map from the Jacobian of the original curve to the Jacobian of an elliptic curve (which is radically parametrizable always). We want to explore the possibility of "lifting" this to a morphism of the curves, basically going in the opposite direction of the functor that sends a map $\phi\colon C_1 \to C_2$ to the induced map $\phi_*\colon J(C_1) \to J(C_2)$. More general cases are of interest as well.

1.4.5 Programming aspects

In addition to the purely mathematical research, we plan to program several algorithms during the project; it is natural to do so given the algorithmic nature of the problem and the possible applications of an efficient solution. We foresee that the currently available Computer Algebra Systems will suffice for our needs. Given our past experience, we intend to make use of the systems MAGMA [BCP97] and SAGE [S+09] for programming our algorithms; the first one has some unique capabilities for algebraic geometry computations and the second one is competent as well, plus it has the desirable features of being free and open source.

1.5 Work plan, time plan, strategies for dissemination of results

1.5.1 Work plan

Our initial goal is to assess the existing knowledge in those areas which we intend to explore as potentially fruitful, as described in Section 1.4: monodromy theory, Galois theory, images of algebraic curves, abelian varieties, etc. This should provide the foundations for more concrete attacks to the problem, which would comprise a second phase of the project. In this regard, the two senior members of the project (see Section 2.2) will work concurrently.

The third member of the project, a Ph. D. student, would dedicate an initial period to the acquisition of general knowledge in Symbolic Computation, Algebraic Geometry and Group Theory (partly through attendance to courses in the Ph. D. program of their enrolment), as needed in order to allow for active involvement in the research. Later, this student, in collaboration with one or two of the other members, will participate in the more concrete aspects which will have arisen during the initial stages of the project. In particular, in Section 1.4.2 we have outlined some steps for the development of an algorithm for detection and computation of tetragonality, which will be carried by the student with the assistance of the other members. An important aspect of this project is the implementation of algorithms (Section 1.4.5), in which the student will take part actively.

1.5.2 Time plan

The following table depicts the estimated timetable for the Ph. D. student.

Task	Year 1	Year 2	Year 3
Acquisition of background knowledge	▬▬▬▬		
Research on tetragonality	▬▬	▬▬▬▬	
Other research		▬▬▬▬	▬▬▬▬
Implementations		▬▬	▬▬▬▬

1.5.3 Strategies for dissemination of results

Regarding the dissemination of results, our strategy is the following:

- We intend to publish the major advances of our research in leading international journals in the areas of Symbolic Computation and Algebraic Geometry.
- We will present our contributions in international conferences on Symbolic Computation and applications, like ISSAC, MEGA, CASC and ACA.
- In relation to the ACA conference, we intend to organize a session on radical parametrizations in one of the future editions of it. This would allow us to further our research, and to make new contacts with other people interested or who could contribute to it.
- The possibility exists to give a course or research seminar (*Spezialvorlesung*) in the Johannes Kepler University in relation to our topics of interest. Given that the audience would probably be graduate and Ph. D. students, a tradeoff between high and low level mathematics needs to be found. Since Dr. Sevilla and Dr. Schicho have full teaching loads programmed until Summer 2011, this course would take place no sooner than Winter 2011/2012.

1.6 Cooperations

One line of collaboration that we intend to pursue is to work with Dr. Rafael Sendra on exploring the extendability of results from rational parametrizations to radical parametrizations. The extensive knowledge of both Dr. Schicho and Dr. Sendra in this area, and their fruitful past collaborations, are strong points in favour.

Dr. Sevilla has collaborated in the past with Dr. Tanush Shaska, working together on hyperelliptic curves with extra involutions [SS07]. Given the expertise of Dr. Shaska in automorphism groups of curves, contacts have been initiated with the intention to work on the approach described in Section 1.4.3.

The results and implementations of Dr. Florian Hess will be fundamental for our research on quotients of curves by automorphisms, as described in Section 1.4.3. We intend to contact him with the perspective of cooperation.

1.7 Potential additional aspects

We would like to highlight the interdisciplinary aspect of the project. Our methods are a mixture of Algebraic Geometry and Computer Algebra/Symbolic Computation with the utilization of Group Theory. As such we think that our research will mainly benefit the first two disciplines. Also, since algebraic curves are fundamental elements in Computer Aided Geometric Design, it is conceivable that some results obtained can be used in Applied Geometry. In particular, for many of the applications using rational parametrizations of algebraic curves, radical parametrizations are equally valid, and our results could greatly enlarge the class of curves parametrizable in an efficient way.

2 Organizational aspects of the project

2.1 Project duration

The proposed duration of the project is of three years. This amount of time will allow for substantial progress in the different paths described in Section 1.4. Furthermore, we intend to involve a Ph. D. student in our project (see below), and this seems a reasonable amount of time for a student to develop a deep understanding of the topic and the techniques that we will use.

2.2 Human resources

The members of the project will be Dr. David Sevilla (project leader), Dr. Josef Schicho, and a Ph. D. student not specified yet.

2.2.1 Scientific qualifications of the scientists involved

Dr. Sevilla is a research scientist in the Symbolic Computation group at the Johann Radon Institute for Computational and Applied Mathematics (RICAM) of the ÖAW (Austrian Academy of Sciences). He completed his Ph. D. in 2004, working on decomposition of polynomials and rational functions; previously he obtained a M. Sc. degree in pure mathematics in the Universidad Complutense of Madrid. His previous research in functional decomposition and automorphism groups of hyperelliptic curves is an additional qualification for this project.

Dr. Schicho is the group leader of the aforementioned Symbolic Computation group. His long experience in Symbolic Computation research, and in particular on parametrizations, makes him an ideal member of the project. Dr. Sevilla and Dr. Schicho have worked together during the last two years and a half, and have achieved some progress in the problem of radical root parametrization, on which they have been collaborating since Dr. Sevilla moved to Austria.

A third member of the team would be a Ph. D. student in Mathematics, for which we have no concrete candidate yet. We have no particular candidate for now. Since the work within the project would involve theoretical mathematics research in symbolic computation, we would prefer candidate would be a person with a degree in Mathematics and some programming knowledge, ideally with Computer Algebra Systems. We expect to incorporate a student as soon as possible after the beginning of the project.

2.2.2 Importance of the project for the career development of the participants

Dr. Sevilla is a junior researcher with working experience in several internationally relevant institutions. At this stage, the leadership of a research project of this scope and the advisory of a Ph. D. student are natural next steps in the development of his scientific career.

The chosen Ph. D. student will have a strong start in their future career in research, given that this project involves both theoretical and more applied aspects of Symbolic Computation research, plus continuous interaction with Computer Algebra systems.

2.3 Work environment

The project will be carried out entirely at the RICAM institute in Linz. This is an ideal environment for several reasons:

- RICAM is a leading institution in computational and applied mathematics. As a mission statement, the institute does basic research in computational and applied mathematics according to highest international standards. One of the very positive traditions and attitudes in the sense of research education and training at RICAM is that supervisors (i.e., the group leaders) concretely and actively share their expertise with younger collegues and stimulate them to independent elaboration of possible developments. In particular, the group leaders tend to encourage young researchers to active cooperations in their early stage. Indeed, the Symbolic Computation group of RICAM provides a very stimulating environment, and the other groups of the institute are a valuable source of scientific exchange.
- The institute is located in the campus of the Johannes Kepler University. One of the research institutes of this university is the Research Institute for Symbolic Computation (RISC) located in the nearby town of Hagenberg. This is an invaluable connection for a research project centered in Symbolic Computation.
- Available infrastructure: the offices of the Symbolic Computation group are completely furnished and equipped with computer infrastructure. Every RICAM member has at their disposal a workstation for extensive numerical experiments and software development, with full access to the latest versions of all major software tools; access to more powerful machines when particularly large computations are required is possible, and laptops for mobility as well. Since RICAM is embedded into the campus of the JKU and shares part of its infrastrucure with the university, RICAM members can access all the relevant facilities, including the very well furnished maths library.

3 Financial aspects

The funds requested are basically those necessary for a Ph. D. position for three years, making a provision for research stays of the student in other institutions. No equipment costs are needed since RICAM will provide all the infraestructure and material that is necessary to all the members of the project.

3.1 Personnel

For the Ph. D. student, we ask for the usual costs of a full-time salary. The oher two members of the project (Dr. Sevilla and Dr. Schicho) are fully supported economically by RICAM.

3.2 Travel support

The estimated travel costs are to be dedicated exclusively to the mobility of the Ph. D. student, since Dr. Sevilla and Dr. Schicho shall use other funds for their own travelling. Also, visits to RICAM from outside guests will be fully supported by RICAM.

We estimate two short research stays in other institutions, which we do not specify yet; at least one of them will probably consist on attendance to a summer school that is adequate for the development of the student, and/or fits with our research topic. For this purpose, we ask for EUR 1.600.

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We use Lie algebra computations (which mostly amount to linear algebra) to decide if a certain algebraic variety associated to the input curve is.Â @article{Schicho2011EffectiveRP, title={Effective radical parametrization of trigonal curves}, author={J. Schicho and D. Sevilla}, journal={ArXiv}, year={2011}, volume={abs/1104.2470} }. J. Schicho, D. Sevilla. Published 2011. Mathematics, Computer Science. ArXiv. Let C be a non-hyperelliptic algebraic curve. Appendix to Part I. Curves and their genus. Topology of nonsingular plane cubics over C; informal discussion of the genus of a curve; topol-ogy, dierential geometry, moduli, number theory, Mordellâ€"Weilâ€"Faltings. Part II. The category of ane varieties.Â Most of the following are textbooks at a graduate level, and some are referred to in the text: W. Fulton, Algebraic curves, Springer. (This is the most down-to-earth and self-contained of. the graduate texts; Ch. All these curves share the property that, beside their geometrical description, they can be given by algebraic equations in the plane equipped with coor-dinates x, y. The equation of the conic sections are of course all quadratic. For the cissoid it reads. y2(2a âˆ' x) = x3 and for the conchoid we have. Although algebraic geometry is a highly developed and thriving eld of mathe-matics, it is notoriously difcult for the beginner to make his way into the subject. There are several texts on an undergraduate level that give an excellent treatment of the classical theory of plane curves, but these do not prepare the student adequately for modern algebraic geometry. On the other hand, most books with a modern ap-proach demand considerable background in algebra and topology, often the equiv-alent of a year or more of graduate study. The aim of these notes is to develop the theory of algebraic curves... Radical parametrization of curves. Examples. All elliptic curves (genus 1) can be expreâˆšssed as y 2 = x3 + ax + b. This can be parametrized as x = t, y = t3 + at + b. Similarly for all hyperelliptic curves (genus â‰¥ 2), y 2 = P(x).Â Radical parametrization of algebraic surfaces. Given an irreducible implicit algebraic surface F (x, y , z) = 0, can we nd a rational parametric representation x(s, t), y (s, t), z(s, t) of it? In general, no: the genera must be 0. So, once more we extend the problem to â€œfunctionsâ€ in s, t expressible by radicals.