

# Outer Jordan Content

Michelle Hummel, Matthew Dahlgren,  
Mahshid Rahnamay-Naeini, Martha Byrne, Ming Gong.

September 21, 2010

## 1 Definition

The **outer Jordan content**  $J_*(E)$  of a set  $E$  in  $\mathbb{R}$  is defined by

$$J_*(E) = \inf \sum_{j=1}^N |I_j|$$

where the inf is taken over every *finite* covering  $E \subset \bigcup_{j=1}^N I_j$ , by intervals  $I_j$ . Compare this with the **exterior measure**  $m_*(E)$  of a set  $E$  where the infimum is taken over all *countable* coverings  $E \subset \bigcup_{j=1}^{\infty} I_j$ .

The **inner Jordan content**  $J^*(E)$  of a set  $E$  in  $\mathbb{R}$  is defined by

$$J^*(E) = \sup \sum_{j=1}^N |I_j|$$

where the sup is taken over every finite set  $E \supset \bigcup_{j=1}^N I_j$ . A set  $E$  is said to be **Jordan measurable** if

$$J_*(E) = J^*(E)$$

## 2 History

The history of the Jordan content (or Peano-Jordan measure) is a small part of a much larger story about math in the late 19th century. The ideas of inner and outer Jordan content were developed independently by the Italian mathematician Giuseppe Peano and the French mathematician Camille Jordan, as generalizations of the concepts of upper and lower Riemann integrals.

Both were dissatisfied with Riemann integration theory when applied in higher dimensions. At the time, it was believed that for rectangles in a partition meeting the boundary of an arbitrary set, the sum of the areas of such

rectangles could be made arbitrarily small. However, this is false, even in the one-dimensional case, as there exist nowhere dense sets which cannot be enclosed by intervals of arbitrarily small length, such as the Smith-Volterra-Cantor set (sometimes called the fat Cantor set.) Peano, in particular, was able to demonstrate a curve which fills the unit square.

Motivated by Cantor's work in set theory, Jordan and Peano set out to provide a more general adaptation of the Riemann integral, and were the first to apply set theory to analysis and integration. Jordan introduced the concept of a Jordan measurable set in 1892. As Jordan was a highly regarded mathematician, his work and endorsement of the application of Cantorian set theory to analysis generated much interest from other prominent mathematicians of the day. Among those interested were Emile Borel and Henri Lebesgue.

Borel took an alternate route, from that of Peano and Jordan, to the question of measurability and stated properties that a measure on sets should have. His ideas, however, introduced some unnecessary conventions in the theory.

Lebesgue later rectified Borel's theory when he consolidated Borel's ideas with those of Jordan to create the notion of Lebesgue measure and the Lebesgue integral.

### 3 Exercise 14

#### 3.1 Prove that $J_*(E) = J_*(\bar{E})$ for every set $E$ .

**First direction ( $\leq$ ):**

Let  $E \subset \mathbb{R}$ . Since  $E \subset \bar{E}$ , if  $\varphi = \{I_j\}_{j=1}^N$  covers  $\bar{E}$ , then  $\varphi$  covers  $E$  as well. Thus,  $J_*(E) \leq J_*(\bar{E})$ .

**Second direction ( $\geq$ ):**

To see  $J_*(E) \geq J_*(\bar{E})$ , it will suffice to show that if  $\varphi = \bigcup_{j=1}^N I_j$  covers  $E$ , then  $\bar{\varphi} = \bigcup_{j=1}^N \bar{I}_j$  covers  $\bar{E}$ . This is sufficient because, by the definition of  $|I_j|$  for  $I_j \subset \mathbb{R}$ ,  $|I_j| = |\bar{I}_j|$ .

Let  $\varphi$  cover  $E$ . Suppose  $\bar{\varphi}$  does not cover  $\bar{E}$ , then  $\exists x \in \bar{E}$ , such that  $x \notin \bar{\varphi}$ . By definition,  $\bar{E} = E \cup E'$  where  $E'$  is the set of all limit points of  $E$ . Thus  $\forall \varepsilon > 0$ ,  $(x - \varepsilon, x + \varepsilon) \cap E \neq \emptyset$ , but  $E \subset \varphi$  which implies  $(x - \varepsilon, x + \varepsilon) \cap \varphi \neq \emptyset$ . Therefore  $x$  is a limit point of  $\varphi$ .

However, since  $\bar{\varphi}$  is the finite union of closed intervals,  $\bar{\varphi}$  is closed and must contain all of its limit points, so  $x \in \bar{\varphi}$ , which is a contradiction. Therefore, if  $E \subset \varphi$ , then  $\bar{E} \subset \bar{\varphi}$ , which implies  $J_*(E) \geq J_*(\bar{E})$ . Thus  $J_*(E) = J_*(\bar{E})$ .  $\square$

### 3.2 Exhibit a countable subset $E \subset [0, 1]$ such that $J_*(E) = 1$ while $m_*(E) = 0$ .

Let  $E = \mathbb{Q} \cap [0, 1]$ . Since  $E$  is dense in  $[0, 1]$ ,  $J_*(E) = J_*(\overline{E}) = J_*([0, 1]) = 1$ . However for each rational  $q_j$ ,  $m_*({q_j}) = 0$  means  $\sum_{j=1}^{\infty} m_*({q_j}) = 0$ .

Moreover, there exists an infinite sequence of sets  $E_n = (\mathbb{Q} \cap [0, 1]) \setminus S_n$  where  $S_n$  is countably infinite, such that  $J_*(E_n) = 1$  while  $m_*(E_n) = 0$ . Let  $S_n = \{\frac{n}{p} : p = n \cdots \infty\}$ . Between any two rationals  $\frac{n}{p}, \frac{n}{p+1} \in E_n$ , there is a subset of rationals,  $Q_p = E_n \cap (\frac{n}{p}, \frac{n}{p+1})$ , that is dense in the interval  $[\frac{n}{p}, \frac{n}{p+1}]$  which means  $E_n$  is dense in  $[0, 1]$ . Hence  $J_*(E_n) = 1$  and  $m_*(E_n) = 0$ .

## 4 Properties of outer Jordan content

In this section we compare the properties of the exterior measure  $m_*(E)$  with the outer Jordan content  $J_*(E)$ .

As stated by Stein and Shakarchi [4], the properties of  $m_*(E)$  are:

1. Monotonicity
2. Countable Sub-additivity
3. Approximation of  $m_*(E)$  by  $m_*(\mathcal{O})$   $E \subset \mathcal{O}$  open
4. Additivity of separated sets
5. Additivity of the countable union of almost disjoint cubes

### 4.1 Monotonicity

If  $E \subseteq F$ , then  $J_*(E) \leq J_*(F)$ .

**Proof:**

If a finite collection of intervals,  $\{I_j\}_{j=1}^N$ , covers  $F$ , then  $E \subset F \subset \bigcup_{j=1}^N I_j$ . This implies that

$$J_*(E) = \inf \sum_{j=1}^N |I_j| \leq \inf \sum_{j=1}^M |J_j| = J_*(F)$$

the infimums are taken over  $E \subset \bigcup_{j=1}^N I_j$  and  $F \subset \bigcup_{j=1}^M J_j$ , respectively. So the outer Jordan content is monotonic with respect to containment.  $\square$

### 4.2 Not countably subadditive

If  $E = \bigcup_{j=1}^{\infty} E_j$  then, in general,  $J_*(E) \not\leq \sum_{j=1}^{\infty} J_*(E_j)$ .

**Proof:**

Let  $E = \mathbb{Q} \cap [0, 1]$ . Since  $\mathbb{Q}$  is dense in  $[0, 1]$ ,  $J_*(E) = J_*(\overline{E}) = J_*([0, 1]) = 1$ . However for each rational  $q_j$ ,  $J_*({q_j}) = 0$  which means  $\sum_{j=1}^{\infty} J_*({q_j}) = 0$ ,

where  $q_j$  is an ordering of  $\mathbb{Q}$ .  $\square$

However, *finite* subadditivity does hold.

### 4.3 Approximation by open sets

If  $E \subset \mathbb{R}$ , then  $J_*(E) = \inf J_*(\mathcal{O})$ , where the infimum is taken over all open sets  $\mathcal{O}$  such that  $E \subset \mathcal{O}$ .

**Proof:**

Since  $J_*$  is monotonic,  $J_*(E) \leq \inf(J_*(\mathcal{O}))$ . It remains to show that  $\inf(J_*(\mathcal{O})) \leq J_*(E)$ .

Fix  $\epsilon > 0$ , and choose cubes  $Q_j$ ,  $E \subset \bigcup_{j=1}^N Q_j$ , such that

$$\sum_{j=1}^N |Q_j| \leq J_*(E) + \frac{\epsilon}{2}.$$

Define an open cube,  $Q_j^0 \supset Q_j$  such that  $|Q_j^0| \leq |Q_j| + \frac{\epsilon}{2^{j+1}}$ . Then  $\mathcal{O} = \bigcup_{j=1}^N Q_j^0$  is open, and

$$J_*(\mathcal{O}) \leq \sum_{j=1}^N J_*(Q_j^0) \tag{1}$$

$$= \sum_{j=1}^N |Q_j^0| \tag{2}$$

$$\leq \sum_{j=1}^N |Q_j| + \frac{\epsilon}{2^{j+1}} \tag{3}$$

$$\leq \sum_{j=1}^N |Q_j| + \frac{\epsilon}{2} \tag{4}$$

$$\leq J_*(E) + \epsilon. \tag{5}$$

Therefore,  $\inf J_*(\mathcal{O}) \leq J_*(E)$ .  $\square$

Note, however, that  $J_*(E)$  cannot be approximated by  $m_*(\mathcal{O})$ . This can be shown by the set  $E = \mathbb{Q} \cap [0, 1]$ .

### 4.4 Additivity of separated sets

If  $E = E_1 \cup E_2$ , and  $d(E_1, E_2) > 0$ , then  $J_*(E) = J_*(E_1) + J_*(E_2)$ .

**Proof:**

First we show that  $J_*(E) \leq J_*(E_1) + J_*(E_2)$ . Assume  $\exists \varphi = \{I_j\}_{j=1}^N$  such that  $E_1 \subset \bigcup_{j=1}^N I_j$  and  $\exists \theta = \{J_j\}_{j=1}^M$  such that  $E_2 \subset \bigcup_{j=1}^M J_j$ . As  $E = E_1 \cup E_2$ ,

$E \subset \Theta \cup \varphi$ . By definition of the outer Jordan content we have

$$J_*(E_1) \geq \sum_{j=1}^N |I_j| - \frac{\varepsilon}{2}$$

$$J_*(E_2) \geq \sum_{j=1}^M |J_j| - \frac{\varepsilon}{2}.$$

By summing both sides of the inequalities we have

$$J_*(E_1) + J_*(E_2) \geq \sum_{j=1}^N |I_j| + \sum_{j=1}^M |J_j| - \varepsilon.$$

Hence,

$$J_*(E_1) + J_*(E_2) \geq J_*(E) - \varepsilon.$$

Since  $\varepsilon$  can be chosen arbitrarily small, we have shown  $J_*(E) \leq J_*(E_1) + J_*(E_2)$ .

For the other direction of the inequality, select  $\delta > 0$  such that  $d(E_1, E_2) > \delta$ . Take a covering  $E \subset \cup_{j=1}^N I_j$  with

$$\sum_{j=1}^N |I_j| \leq J_*(E) + \varepsilon.$$

Then divide the intervals  $I_j$  into subintervals with the length less than  $\delta$ . Each new interval can intersect at most one of the two sets  $E_1, E_2$ . Let the sets  $J_1$  and  $J_2$  contain the indices of the intervals in  $I_j$  which intersect  $E_1$  and  $E_2$ , respectively. Then  $J_1 \cap J_2$  is empty and we have

$$E_1 \subset \cup_{j \in J_1} I_j$$

$$E_2 \subset \cup_{j \in J_2} I_j.$$

Therefore,

$$J_*(E_1) + J_*(E_2) \leq \sum_{j \in J_1} |I_j| + \sum_{j \in J_2} |I_j| \leq \sum_{j=1}^N |I_j|$$

which means,

$$J_*(E_1) + J_*(E_2) \leq J_*(E) + \varepsilon.$$

Since  $\varepsilon$  can be arbitrarily small, the proof is complete.  $\square$

## 4.5 Countable unions of almost disjoint cubes are not additive

If a set  $E$  is the countable union of almost disjoint cubes  $E = \bigcup_{j=1}^{\infty} Q_j$ , then in general,  $J_*(E) \neq \sum_{j=1}^{\infty} |Q_j|$ .

**Proof:**

Consider  $E = \mathbb{Q} \cap [0, 1]$ .  $\square$

## References

- [1] Frank Burk, *Lebesgue Measure and Integration: An Introduction* (New York: John Wiley & Sons Inc., 1998), 20-26.
- [2] Jewgeni H. Dshalalow, *Real analysis: an introduction to the theory of real functions and integration* (Boca Raton, Florida: CRC Press LLC, 2001), 221-222.
- [3] Grattan-Guinness and H. J. M. Bos, *From the Calculus to Set Theory, 1630-1910: An Introductory History* (Princeton, New Jersey: Princeton University Press, 1980), 164-180.
- [4] Elias Stein, Rami Shakarchi *Real Analysis: Measure Theory, Integration, and Hilbert Spaces* (Princeton, New Jersey: Princeton University Press, 2005), 13-15, 41.

Outer Jordan Content. Michelle Hummel, Matthew Dahlgren, Mahshid Rahnamay-Naeini, Martha Byrne, Ming Gong. September 21, 2010. 1 Definition The outer Jordan content  $J^*(E)$  of a set  $E$  in  $\mathbb{R}^n$  is defined by  $J^*(E) = \inf \sum_{j=1}^N |I_j|$  where the inf is taken over every finite covering  $E \subset \bigcup_{j=1}^N I_j$ , by intervals  $I_j$ . Compare this with the exterior measure  $m^*(E)$  of a set  $E$  where the infimum is taken over all countable coverings  $E \subset \bigcup_{j=1}^{\infty} I_j$ . The inner Jordan content  $J_*(E)$  of a set  $E$  in  $\mathbb{R}^n$  is defined by  $J_*(E) = \sup \sum_{j=1}^N |I_j|$ . Skip to content. Categories. Search for anything. Development. Web Development Data Science Mobile Development Programming Languages Game Development Database Design & Development Software Testing Software Engineering Development Tools No-Code Development. Business. Course content. 3 sections 59 lectures 4h 57m total length. Expand all sections. Introduction 12 lectures 1hr 20min. Element. Preview 05:18. 02 - Development of the Straight Line. The volume or Jordan content of a Jordan measurable set  $X \subset \mathbb{R}^n$  will be denoted by  $V(X)$  and the outer Jordan content of a bounded set  $Y$  by  $J^*(Y)$ . By a Jordan partition of  $X \subset \mathbb{R}^n$  we mean a finite collection of pairwise non-overlapping sets in  $\mathbb{R}^n$  the union of which is  $X$ . The set of all Jordan partitions of  $X \subset \mathbb{R}^n$  will be denoted by  $\mathcal{P}(X)$ . By the norm of a Jordan partition  $\mathcal{P} \in \mathcal{P}(X)$ .