

A Riemann-Lagrange geometrization for metrical Multi-Time Lagrange Spaces

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Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. In this paper we construct a geometrization on the 1-jet fiber bundle $J^1(T, M)$ for the multi-time quadratic Lagrangian function

$$L = h^{\alpha\beta}(t)g_{ij}(t, x)x_\alpha^i x_\beta^j + U_{(i)}^{(\alpha)}(t, x)x_\alpha^i + F(t, x).$$

Our geometrization includes a nonlinear connection Γ , a generalized Cartan canonical Γ -linear connection CT together with its d -torsions and d -curvatures, naturally provided by the given multi-time quadratic Lagrangian function L .

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1 Metrical multi-time Lagrange spaces

It is important to note that quadratic multi-time Lagrangians dominates most scientific domains. We can remind only the theory of elasticity [19], the dynamics of ideal fluids, the magnetohydrodynamics [6], [7], the theory of bosonic strings [5] or the multi-time evolution (p-flow) of some physical or economical phenomena [21, 22, 23, 24, 25, 26]. This fact emphasizes the importance of the geometrization of quadratic multi-time Lagrangians. In conclusion, a Riemann-Lagrange geometry on 1-jet spaces was required. Such a geometry was initially developed by Saunders [20] and Asanov [2], and continued, in a Miron's approach [10], by Udriște ([21]-[26]) and Neagu ([13], [16]).

In the sequel, let us fix $h = (h_{\alpha\beta}(t^\gamma))$ a semi-Riemannian metric on the temporal manifold T and let $g = (g_{ij}(t^\gamma, x^k, x_\gamma^k))$ be a symmetric d -tensor on $E = J^1(T, M)$, of rank n and having a constant signature.

Generally, a smooth multi-time Lagrangian function

$$(1.1.1) \quad L : E \rightarrow \mathbb{R}, \quad E \ni (t^\alpha, x^i, x_\alpha^i) \rightarrow L(t^\alpha, x^i, x_\alpha^i) \in \mathbb{R},$$

produces a *fundamental vertical metrical d-tensor*

$$(1.1.2) \quad G_{(i)(j)}^{(\alpha)(\beta)} = \frac{1}{2} \frac{\partial^2 L}{\partial x_\alpha^i \partial x_\beta^j},$$

where $i, j = 1, \dots, n$ and $\alpha, \beta = 1, \dots, p$.

Definition 1.1. A multi-time Lagrangian function $L : E \rightarrow \mathbb{R}$, having the fundamental vertical metrical d-tensor of the form

$$(1.1.3) \quad G_{(i)(j)}^{(\alpha)(\beta)}(t^\gamma, x^k, x_\gamma^k) = h^{\alpha\beta}(t^\gamma) g_{ij}(t^\gamma, x^k, x_\gamma^k),$$

is called a **Kronecker h-regular multi-time Lagrangian function**.

In this context, we can introduce the following important concept:

Definition 1.2. A pair $ML_p^n = (J^1(T, M), L)$, $p = \dim T$, $n = \dim M$, consisting of the 1-jet fibre bundle and a Kronecker h-regular multi-time Lagrangian function $L : J^1(T, M) \rightarrow \mathbb{R}$, is called a **multi-time Lagrange space**.

Remark 1.3. *i)* In the particular case $(T, h) = (\mathbb{R}, \delta)$, a multi-time Lagrange space is called a **relativistic rheonomic Lagrange space** and is denoted by

$$RL^n = (J^1(\mathbb{R}, M), L).$$

For more details about the relativistic rheonomic Lagrangian geometry, the reader may consult [15].

ii) If the temporal manifold T is 1-dimensional one, then, via a temporal reparametrization, we have

$$J^1(T, M) \equiv J^1(\mathbb{R}, M).$$

In other words, a multi-time Lagrange space, having $\dim T = 1$, is a reparametrized relativistic rheonomic Lagrange space.

Example 1.4. Let us suppose that the spatial manifold M is also endowed with a semi-Riemannian metric $g = (g_{ij}(x))$. Then, the multi-time Lagrangian function

$$(1.1.4) \quad L_1 : E \rightarrow \mathbb{R}, \quad L_1 = h^{\alpha\beta}(t) g_{ij}(x) x_\alpha^i x_\beta^j$$

is a Kronecker h-regular one. It follows that the pair

$$\mathcal{BSML}_p^n = (J^1(T, M), L_1)$$

is a multi-time Lagrange space. It is important to note that the multi-time Lagrangian $\mathcal{L}_1 = L_1 \sqrt{|\hbar|}$ is exactly the "energy" Lagrangian, whose extremals are ultra-harmonic maps between the semi-Riemannian manifolds (T, h) and (M, g) [4]. At the same time, the multi-time Lagrangian that governs the physical theory of bosonic strings is of type \mathcal{L}_1 [6].

Example 1.5. (geometric dynamics [21, 22, 23, 24, 25, 26]) Let us start with a p -flow described by the completely integrable PDES system

$$\frac{\partial x^i}{\partial t^\alpha} = X_\alpha^i(t, x(t)), \quad i = 1, \dots, n; \quad \alpha = 1, \dots, p.$$

This system and the semi-Riemannian metrics $h^{\alpha\beta}(t)$ and $g_{ij}(t, x)$ determine the quadratic Lagrangian function

$$(1.1.5) \quad L_2 : E \rightarrow \mathbb{R}, \quad L_2 = h^{\alpha\beta}(t)g_{ij}(t, x)(x_\alpha^i - X_\alpha^i(t, x(t)))(x_\beta^j - X_\beta^j(t, x(t))),$$

which is Kronecker h -regular. If the metrics h and g are positive definite, then this is the least squares Lagrangian. Also, we remark that any PDEs system can be replaced with a p -flow (multi-time evolution), and consequently it produces a Lagrange-Hamilton problem via any quadratic Lagrange function of preceding form.

Example 1.6. In the above notations, taking $U_{(i)}^{(\alpha)}(t, x)$ a d -tensor field on E and $F : T \times M \rightarrow \mathbb{R}$ a smooth function, the quadratic multi-time Lagrangian function

$$(1.1.6) \quad L_2 : E \rightarrow \mathbb{R}, \quad L_2 = h^{\alpha\beta}(t)g_{ij}(x)x_\alpha^i x_\beta^j + U_{(i)}^{(\alpha)}(t, x)x_\alpha^i + F(t, x),$$

is also a Kronecker h -regular one. The multi-time Lagrange space

$$\mathcal{EDML}_p^n = (J^1(T, M), L_2)$$

is called the **autonomous multi-time Lagrange space of electrodynamics**. This is because, in the particular case $(T, h) = (\mathbb{R}, \delta)$, the space \mathcal{EDML}_1^n naturally generalizes the classical Lagrange space of electrodynamics, that governs the movement law of a particle placed concomitently into a gravitational field and an electromagnetic one. From physical point of view, the semi-Riemannian metric $h_{\alpha\beta}(t)$ (resp. $g_{ij}(x)$) represents the **gravitational potentials** of the manifold T (resp. M), the d -tensor $U_{(i)}^{(\alpha)}(t, x)$ plays the role of the **electromagnetic potentials** which produce a gyroscopic force, and F is a **potential function**. The non-dynamical character of the spatial gravitational potentials $g_{ij}(x)$ motivates us to use the term "autonomous".

Example 1.7. More general, if we consider the symmetrical d -tensor $g_{ij}(t, x)$ on E , of rank n and having a constant signature on E , we can define the Kronecker h -regular multi-time Lagrangian function

$$(1.1.7) \quad L_3 : E \rightarrow \mathbb{R}, \quad L_3 = h^{\alpha\beta}(t)g_{ij}(t, x)x_\alpha^i x_\beta^j + U_{(i)}^{(\alpha)}(t, x)x_\alpha^i + F(t, x).$$

The multi-time Lagrange space

$$\mathcal{NEDML}_p^n = (J^1(T, M), L_3)$$

is called the **non-autonomous multi-time Lagrange space of electrodynamics**. We use the term "non-autonomous", in order to emphasize the dynamical character of spatial gravitational potentials $g_{ij}(t, x)$, i.e., their dependence of the temporal coordinates t^γ .

An important role and, at the same time, an obstruction in the subsequent development of the theory of the multi-time Lagrange spaces, is played by the following theorem, proved in [12]:

Theorem 1.8. *(of characterization of multi-time Lagrange spaces)*

If $p = \dim T \geq 2$, then the following statements are equivalent:

i) L is a Kronecker h -regular Lagrangian function on $J^1(T, M)$.

ii) The multi-time Lagrangian function L reduces to a multi-time Lagrangian function of non-autonomous electrodynamic type, that is

$$L = h^{\alpha\beta}(t)g_{ij}(t, x)x_\alpha^i x_\beta^j + U_{(i)}^{(\alpha)}(t, x)x_\alpha^i + F(t, x).$$

Corollary 1.9. *The fundamental vertical metrical d -tensor of an arbitrary Kronecker h -regular multi-time Lagrangian function L is of the form*

$$(1.1.8) \quad G_{(i)(j)}^{(\alpha)(\beta)} = \frac{1}{2} \frac{\partial^2 L}{\partial x_\alpha^i \partial x_\beta^j} = \begin{cases} h^{11}(t)g_{ij}(t, x^k, y^k), & p = \dim T = 1 \\ h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k), & p = \dim T \geq 2. \end{cases}$$

Remark 1.10. i) It is obvious that the preceding theorem is an obstruction in the development of a fertile geometrical theory for the multi-time Lagrange spaces. This obstruction was surpassed in the paper [13], by introducing the more general notion of a **generalized multi-time Lagrange space**. The generalized multi-time Riemann-Lagrange geometry on $J^1(T, M)$ will be constructed using only a Kronecker h -regular vertical metrical d -tensor $G_{(i)(j)}^{(\alpha)(\beta)}$ and a nonlinear connection Γ , "a priori" given on the 1-jet space $J^1(T, M)$.

ii) In the case $p = \dim T \geq 2$, the preceding theorem obliges us to continue our geometrical study of the multi-time Lagrange spaces, directing our attention upon the non-autonomous multi-time Lagrange spaces of electrodynamics.

Let $ML_p^n = (J^1(T, M), L)$, where $\dim T = p$, $\dim M = n$, be a multi-time Lagrange space whose fundamental vertical metrical d -tensor metric is

$$G_{(i)(j)}^{(\alpha)(\beta)} = \frac{1}{2} \frac{\partial^2 L}{\partial x_\alpha^i \partial x_\beta^j} = \begin{cases} h^{11}(t)g_{ij}(t, x^k, y^k), & p = 1 \\ h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k), & p \geq 2. \end{cases}$$

Supposing that the semi-Riemannian temporal manifold (T, h) is compact and orientable, by integration on the manifold T , we can define the *energy functional* associated to the multi-time Lagrange function L , taking

$$\mathcal{E}_L : C^\infty(T, M) \rightarrow \mathbb{R}, \quad \mathcal{E}_L(f) = \int_T L(t^\alpha, x^i, x_\alpha^i) \sqrt{|h|} dt^1 \wedge dt^2 \wedge \dots \wedge dt^p,$$

where the smooth map f is locally expressed by $(t^\alpha) \rightarrow (x^i(t^\alpha))$ and $x_\alpha^i = \frac{\partial x^i}{\partial t^\alpha}$.

The extremals of the energy functional \mathcal{E}_L verify the Euler-Lagrange PDEs

$$(1.1.9) \quad 2G_{(i)(j)}^{(\alpha)(\beta)} x_{\alpha\beta}^j + \frac{\partial^2 L}{\partial x^j \partial x_\alpha^i} x_\alpha^j - \frac{\partial L}{\partial x^i} + \frac{\partial^2 L}{\partial t^\alpha \partial x_\alpha^i} + \frac{\partial L}{\partial x_\alpha^i} H_{\alpha\gamma}^\gamma = 0,$$

where $x_{\alpha\beta}^j = \frac{\partial^2 x^j}{\partial t^\alpha \partial t^\beta}$ and $H_{\alpha\beta}^\gamma$ are the Christoffel symbols of the semi-Riemannian temporal metric $h_{\alpha\beta}$.

Taking into account the Kronecker h -regularity of the Lagrangian function L , it is possible to rearrange the Euler-Lagrange equations of the Lagrangian $\mathcal{L} = L\sqrt{|h|}$ in the following *generalized Poisson form (ultra-hyperbolic partial differential equations)*:

$$(1.1.10) \quad \mathbb{E}_h x^k + 2\mathcal{G}^k(t^\mu, x^m, x_\mu^m) = 0,$$

where

$$\mathbb{E}_h x^k = h^{\alpha\beta} \{x_{\alpha\beta}^k - H_{\alpha\beta}^\gamma x_\gamma^k\},$$

$$2\mathcal{G}^k = \frac{g^{ki}}{2} \left\{ \frac{\partial^2 L}{\partial x^j \partial x_\alpha^i} x_\alpha^j - \frac{\partial L}{\partial x^i} + \frac{\partial^2 L}{\partial t^\alpha \partial x_\alpha^i} + \frac{\partial L}{\partial x_\alpha^i} H_{\alpha\gamma}^\gamma + 2g_{ij} h^{\alpha\beta} H_{\alpha\beta}^\gamma x_\gamma^j \right\}.$$

Proposition 1.11. *i) The geometrical object $\mathcal{G} = (\mathcal{G}^r)$ is a multi-time dependent spatial h -spray.*

ii) Moreover, the spatial h -spray $\mathcal{G} = (\mathcal{G}^l)$ is the h -trace of a multi-time dependent spatial spray $G = (G_{(\alpha)\beta}^{(i)})$, that is $\mathcal{G}^l = h^{\alpha\beta} G_{(\alpha)\beta}^{(l)}$.

Proof. The proof of this proposition is given in [12]. \square

Following previous reasonings and the preceding result, we can regard the equations (1.1.10) as being the equations of the ultra-harmonic maps of a multi-time dependent spray.

Theorem 1.12. *The extremals of the energy functional \mathcal{E}_L attached to the Kronecker h -regular Lagrangian function L are ultra-harmonic maps on $J^1(T, M)$ of the multi-time dependent spray (H, G) defined by the temporal components*

$$H_{(\alpha)\beta}^{(i)} = \begin{cases} -\frac{1}{2} H_{11}^1(t) y^i, & p = 1 \\ -\frac{1}{2} H_{\alpha\beta}^\gamma x_\gamma^i, & p \geq 2 \end{cases}$$

and the local spatial components $G_{(\alpha)\beta}^{(i)} =$

$$= \begin{cases} \frac{h_{11} g^{ik}}{4} \left[\frac{\partial^2 L}{\partial x^j \partial y^k} y^j - \frac{\partial L}{\partial x^k} + \frac{\partial^2 L}{\partial t \partial y^k} + \frac{\partial L}{\partial x^k} H_{11}^1 + 2h^{11} H_{11}^1 g_{kl} y^l \right], & p = 1 \\ \frac{1}{2} \Gamma_{jk}^i x_\alpha^j x_\beta^k + T_{(\alpha)\beta}^{(i)}, & p \geq 2, \end{cases}$$

where $p = \dim T$.

Definition 1.13. *The multi-time dependent spray (H, G) constructed in the preceding Theorem is called the canonical multi-time spray attached to the multi-time Lagrange space ML_p^n .*

In the sequel, by local computations, the canonical multi-time spray (H, G) of the multi-time Lagrange space ML_p^n induces naturally a nonlinear connection Γ on $J^1(T, M)$.

Theorem 1.14. The canonical nonlinear connection

$$\Gamma = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)})$$

of the multi-time Lagrange space ML_p^n is defined by the temporal components

$$(1.1.11) \quad M_{(\alpha)\beta}^{(i)} = 2H_{(\alpha)\beta}^{(i)} = \begin{cases} -H_{11}^1 y^i, & p = 1 \\ -H_{\alpha\beta}^\gamma x_\gamma^i, & p \geq 2, \end{cases}$$

and the spatial components

$$(1.1.12) \quad N_{(\alpha)j}^{(i)} = \frac{\partial \mathcal{G}^i}{\partial x_\gamma^j} h_{\alpha\gamma} = \begin{cases} h_{11} \frac{\partial \mathcal{G}^i}{\partial y^j}, & p = 1 \\ \Gamma_{jk}^i x_\alpha^k + \frac{g^{ik}}{2} \frac{\partial g_{jk}}{\partial t^\alpha} + \frac{g^{ik}}{4} h_{\alpha\gamma} U_{(k)j}^{(\gamma)}, & p \geq 2, \end{cases}$$

where $\mathcal{G}^i = h^{\alpha\beta} G_{(\alpha)\beta}^{(i)}$.

Remark 1.15. In the particular case $(T, h) = (\mathbb{R}, \delta)$, the canonical nonlinear connection $\Gamma = (0, N_{(1)j}^{(i)})$ of the relativistic rheonomic Lagrange space $RL^n = (J^1(\mathbb{R}, M), L)$ generalizes naturally the canonical nonlinear connection of the classical rheonomic Lagrange space $L^n = (\mathbb{R} \times \mathbf{TM}, L)$ [10].

2 Generalized Cartan canonical connection $C\Gamma$ of a metrical multi-time Lagrange space

Now, let us consider that $ML_p^n = (J^1(T, M), L)$ is a multi-time Lagrange space, whose fundamental vertical metrical d-tensor is

$$G_{(i)(j)}^{(\alpha)(\beta)} = \frac{1}{2} \frac{\partial^2 L}{\partial x_\alpha^i \partial x_\beta^j} = \begin{cases} h^{11}(t) g_{ij}(t, x^k, y^k), & p = 1 \\ h^{\alpha\beta}(t^\gamma) g_{ij}(t^\gamma, x^k), & p \geq 2. \end{cases}$$

Let $\Gamma = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)})$ be the canonical nonlinear connection of the multi-time Lagrange space ML_p^n .

The main result of this Section is the Theorem of existence and uniqueness of the *generalized Cartan canonical connection* $C\Gamma$, which allowed us to develop in the paper [14] the *multi-time Riemann-Lagrange geometry of physical fields*, theory that represents a natural generalization of the classical field theories (the *Finslerian theory* [1], [2] and the *ordinary Lagrangian theory* [10]).

Theorem 2.1. (the generalized Cartan canonical connection)

On the multi-time Lagrange space $ML_p^n = (J^1(T, M), L)$, endowed with the canonical nonlinear connection Γ , there is a unique h -normal Γ -linear connection

$$CT = (H_{\alpha\beta}^\gamma, G_{j\gamma}^k, L_{jk}^i, C_{j(k)}^{i(\gamma)}),$$

having the metrical properties:

$$i) g_{ij|k} = 0, \quad g_{ij|_{(k)}^{(\gamma)}} = 0,$$

$$ii) G_{j\gamma}^k = \frac{g^{ki}}{2} \frac{\delta g_{ij}}{\delta t^\gamma}, \quad L_{ij}^k = L_{ji}^k, \quad C_{j(k)}^{i(\gamma)} = C_{k(j)}^{i(\gamma)},$$

where " $|_\alpha$ ", " $|_i$ " and " $|_{(i)}^{(\alpha)}$ " are the local covariant derivatives of the h -normal Γ -linear connection CT .

Proof. Let $CT = (\bar{G}_{\alpha\beta}^\gamma, G_{j\gamma}^k, L_{jk}^i, C_{j(k)}^{i(\gamma)})$ be an h -normal Γ -linear connection, whose local coefficients are defined by the relations $\bar{G}_{\alpha\beta}^\gamma = H_{\alpha\beta}^\gamma$, $G_{j\gamma}^k = \frac{g^{ki}}{2} \frac{\delta g_{ij}}{\delta t^\gamma}$ and

$$(2.2.1) \quad \begin{aligned} L_{jk}^i &= \frac{g^{im}}{2} \left(\frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{km}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m} \right), \\ C_{j(k)}^{i(\gamma)} &= \frac{g^{im}}{2} \left(\frac{\partial g_{jm}}{\partial x_\gamma^k} + \frac{\partial g_{km}}{\partial x_\gamma^j} - \frac{\partial g_{jk}}{\partial x_\gamma^m} \right). \end{aligned}$$

Taking into account the local expressions of the local covariant derivatives induced by the connection Γ , by a local calculation, we deduce that CT satisfies the conditions *i)* and *ii)*.

Conversely, let us consider an h -normal Γ -linear connection

$$\tilde{C}\Gamma = (\tilde{G}_{\alpha\beta}^\gamma, \tilde{G}_{j\gamma}^k, \tilde{L}_{jk}^i, \tilde{C}_{j(k)}^{i(\gamma)})$$

which satisfies the metrical conditions *i)* and *ii)*. In this context, we have

$$\tilde{G}_{\alpha\beta}^\gamma = H_{\alpha\beta}^\gamma, \quad \tilde{G}_{j\gamma}^k = \frac{g^{ki}}{2} \frac{\delta g_{ij}}{\delta t^\gamma}.$$

Moreover, the condition $g_{ij|k} = 0$ is equivalent to

$$\frac{\delta g_{ij}}{\delta x^k} = g_{mj} \tilde{L}_{ik}^m + g_{im} \tilde{L}_{jk}^m.$$

Applying now a Christoffel process to the indices $\{i, j, k\}$, we find

$$\tilde{L}_{jk}^i = \frac{g^{im}}{2} \left(\frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{km}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m} \right).$$

By analogy, using the relations $C_{j(k)}^{i(\gamma)} = C_{k(j)}^{i(\gamma)}$ and $g_{ij|_{(k)}^{(\gamma)}} = 0$ and using also a Christoffel process applied to the indices $\{i, j, k\}$, we obtain

$$\tilde{C}_{j(k)}^{i(\gamma)} = \frac{g^{im}}{2} \left(\frac{\partial g_{jm}}{\partial x_\gamma^k} + \frac{\partial g_{km}}{\partial x_\gamma^j} - \frac{\partial g_{jk}}{\partial x_\gamma^m} \right).$$

In conclusion, the uniqueness of the generalized Cartan canonical connection CT is clear. \square

Remark 2.2. *i) Replacing the canonical nonlinear connection Γ with an arbitrary nonlinear connection, the preceding Theorem holds good.*

ii) In the particular case $(T, h) = (\mathbb{R}, \delta)$, the generalized δ -normal Γ -linear Cartan connection associated to the relativistic rheonomic Lagrange space

$$RL^n = (J^1(\mathbb{R}, M), L)$$

generalizes naturally the canonical Cartan connection of a classical rheonomic Lagrange space $L^n = (\mathbb{R} \times \mathbf{T}M, L)$, constructed in [10].

iii) The generalized Cartan canonical connection of the multi-time Lagrange space ML_p^n verifies also the metrical properties

$$h_{\alpha\beta/\gamma} = h_{\alpha\beta|k} = h_{\alpha\beta|k}^{(\gamma)} = 0, \quad g_{ij/\gamma} = 0.$$

iv) In the case $p = \dim T \geq 2$, the coefficients of the generalized Cartan canonical connection of the multi-time Lagrange space ML_p^n reduce to

$$\bar{G}_{\alpha\beta}^\gamma = H_{\alpha\beta}^\gamma, \quad G_{j\gamma}^k = \frac{g^{ki}}{2} \frac{\partial g_{ij}}{\partial t^\gamma}, \quad L_{jk}^i = \Gamma_{jk}^i, \quad C_{j(k)}^{i(\gamma)} = 0.$$

3 Local d-torsions and d-curvatures of $C\Gamma$

Applying the formulas that determine the local d-torsions and d-curvatures of an h -normal Γ -linear connection $\nabla\Gamma$ (see [16]) to the generalized Cartan canonical connection $C\Gamma$, we obtain the following results:

Theorem 3.1. *The torsion d-tensor \mathbf{T} of the generalized Cartan canonical connection $C\Gamma$ of the multi-time Lagrange space ML_p^n is determined by the local components*

	h_T		h_M		v	
	$p = 1$	$p \geq 2$	$p = 1$	$p \geq 2$	$p = 1$	$p \geq 2$
$h_T h_T$	0	0	0	0	0	$R_{(\mu)\alpha\beta}^{(m)}$
$h_M h_T$	0	0	T_{1j}^m	$T_{\alpha j}^m$	$R_{(1)1j}^{(m)}$	$R_{(\mu)\alpha j}^{(m)}$
$h_M h_M$	0	0	0	0	$R_{(1)ij}^{(m)}$	$R_{(\mu)ij}^{(m)}$
$v h_T$	0	0	0	0	$P_{(1)1(j)}^{(m) (1)}$	$P_{(\mu)\alpha(j)}^{(m) (\beta)}$
$v h_M$	0	0	$P_{i(j)}^{m(1)}$	0	$P_{(1)i(j)}^{(m) (1)}$	0
vv	0	0	0	0	0	0

where,

i) for $p = \dim T = 1$ we have

$$T_{1j}^m = -G_{j1}^m, \quad P_{i(j)}^{m(1)} = C_{i(j)}^{m(1)}, \quad P_{(1)1(j)}^{(m) (1)} = -G_{j1}^m,$$

$$P_{(1)i(j)}^{(m) (1)} = \frac{\partial N_{(1)i}^{(m)}}{\partial y^j} - L_{ji}^m, \quad R_{(1)ij}^{(m)} = \frac{\delta N_{(1)i}^{(m)}}{\delta x^j} - \frac{\delta N_{(1)j}^{(m)}}{\delta x^i},$$

$$R_{(1)1j}^{(m)} = -\frac{\partial N_{(1)j}^{(m)}}{\partial t} + H_{11}^1 \left[N_{(1)j}^{(m)} - \frac{\partial N_{(1)j}^{(m)}}{\partial y^k} y^k \right];$$

ii) for $p = \dim T \geq 2$, denoting

$$\begin{aligned} F_{i(\mu)}^m &= \frac{g^{mp}}{2} \left[\frac{\partial g_{pi}}{\partial t^\mu} + \frac{1}{2} h_{\mu\beta} U_{(p)i}^{(\beta)} \right], \\ H_{\mu\alpha\beta}^\gamma &= \frac{\partial H_{\mu\alpha}^\gamma}{\partial t^\beta} - \frac{\partial H_{\mu\beta}^\gamma}{\partial t^\alpha} + H_{\mu\alpha}^\eta H_{\eta\beta}^\gamma - H_{\mu\beta}^\eta H_{\eta\alpha}^\gamma, \\ r_{pij}^m &= \frac{\partial \Gamma_{pi}^m}{\partial x^j} - \frac{\partial \Gamma_{pj}^m}{\partial x^i} + \Gamma_{pi}^k \Gamma_{kj}^m - \Gamma_{pj}^k \Gamma_{ki}^m, \end{aligned}$$

we have

$$\begin{aligned} T_{\alpha j}^m &= -G_{j\alpha}^m, \quad P_{(\mu)\alpha(j)}^{m(\beta)} = -\delta_\gamma^\beta G_{j\alpha}^m, \quad R_{(\mu)\alpha(j)}^{(m)} = -H_{\mu\alpha\beta}^\gamma x_\gamma^m, \\ R_{(\mu)\alpha j}^{(m)} &= -\frac{\partial N_{(\mu)j}^{(m)}}{\partial t^\alpha} + \frac{g^{mk}}{2} H_{\mu\alpha}^\beta \left[\frac{\partial g_{jk}}{\partial t^\beta} + \frac{h_{\beta\gamma}}{2} U_{(k)j}^{(\gamma)} \right], \\ R_{(\mu)ij}^{(m)} &= r_{ijk}^m x_\mu^k + \left[F_{i(\mu)|j}^m - F_{j(\mu)|i}^m \right]; \end{aligned}$$

Theorem 3.2. *The curvature d-tensor \mathbf{R} of the generalized Cartan canonical connection $C\Gamma$ of the multi-time Lagrange space ML_p^n is determined by the local components*

	h_T		h_M		v	
	$p = 1$	$p \geq 2$	$p = 1$	$p \geq 2$	$p = 1$	$p \geq 2$
$h_T h_T$	0	$H_{\eta\beta\gamma}^\alpha$	0	$R_{i\beta\gamma}^l$	0	$R_{(\eta)(i)\beta\gamma}^{(l)(\alpha)}$
$h_M h_T$	0	0	R_{i1k}^l	$R_{i\beta k}^l$	$R_{(1)(i)1k}^{(l)(1)} = R_{i1k}^l$	$R_{(\eta)(i)\beta k}^{(l)(\alpha)}$
$h_M h_M$	0	0	R_{ijk}^l	R_{ijk}^l	$R_{(1)(i)jk}^{(l)(1)} = R_{ijk}^l$	$R_{(\eta)(i)jk}^{(l)(\alpha)}$
vh_T	0	0	$P_{i1(k)}^{l(1)}$	0	$P_{(1)(i)1(k)}^{(l)(1)(1)} = P_{i1(k)}^{l(1)}$	0
vh_M	0	0	$P_{ij(k)}^{l(1)}$	0	$P_{(1)(i)j(k)}^{(l)(1)(1)} = P_{ij(k)}^{l(1)}$	0
vv	0	0	$S_{i(j)(k)}^{l(1)(1)}$	0	$S_{(1)(i)(j)(k)}^{(l)(1)(1)(1)} = S_{i(j)(k)}^{l(1)(1)}$	0

where $R_{(\eta)(i)\beta\gamma}^{(l)(\alpha)} = \delta_\eta^\alpha R_{i\beta\gamma}^l + \delta_i^l H_{\eta\beta\gamma}^\alpha$, $R_{(\eta)(i)\beta k}^{(l)(\alpha)} = \delta_\eta^\alpha R_{i\beta k}^l$, $R_{(\eta)(i)jk}^{(l)(\alpha)} = \delta_\eta^\alpha R_{ijk}^l$ and

i) for $p = \dim T = 1$ we have

$$\begin{aligned} R_{i1k}^l &= \frac{\delta G_{i1}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta t} + G_{i1}^m L_{mk}^l - L_{ik}^m G_{m1}^l + C_{i(m)}^{l(1)} R_{(1)1k}^{(m)}, \\ R_{ijk}^l &= \frac{\delta L_{ij}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta x^j} + L_{ij}^m L_{mk}^l - L_{ik}^m L_{mj}^l + C_{i(m)}^{l(1)} R_{(1)jk}^{(m)}, \\ P_{i1(k)}^{l(1)} &= \frac{\partial G_{i1}^l}{\partial y^k} - C_{i(k)/1}^{l(1)} + C_{i(m)}^{l(1)} P_{(1)1(k)}^{(m)(1)}, \\ P_{ij(k)}^{l(1)} &= \frac{\partial L_{ij}^l}{\partial y^k} - C_{i(k)|j}^{l(1)} + C_{i(m)}^{l(1)} P_{(1)j(k)}^{(m)(1)}, \end{aligned}$$

$$S_{i(j)(k)}^{l(1)(1)} = \frac{\partial C_{i(j)}^{l(1)}}{\partial y^k} - \frac{\partial C_{i(k)}^{l(1)}}{\partial y^j} + C_{i(j)}^{m(1)} C_{m(k)}^{l(1)} - C_{i(k)}^{m(1)} C_{m(j)}^{l(1)} ;$$

ii) for $p = \dim T \geq 2$ we have

$$H_{\eta\beta\gamma}^\alpha = \frac{\partial H_{\eta\beta}^\alpha}{\partial t^\gamma} - \frac{\partial H_{\eta\gamma}^\alpha}{\partial t^\beta} + H_{\eta\beta}^\mu H_{\mu\gamma}^\alpha - H_{\eta\gamma}^\mu H_{\mu\beta}^\alpha ,$$

$$R_{i\beta\gamma}^l = \frac{\delta G_{i\beta}^l}{\delta t^\gamma} - \frac{\delta G_{i\gamma}^l}{\delta t^\beta} + G_{i\beta}^m G_{m\gamma}^l - G_{i\gamma}^m G_{m\beta}^l ,$$

$$R_{i\beta k}^l = \frac{\delta G_{i\beta}^l}{\delta x^k} - \frac{\delta \Gamma_{ik}^l}{\delta t^\beta} + G_{i\beta}^m \Gamma_{mk}^l - \Gamma_{ik}^m G_{m\beta}^l ,$$

$$R_{ijk}^l = r_{ijk}^l = \frac{\partial \Gamma_{ij}^l}{\partial x^k} - \frac{\partial \Gamma_{ik}^l}{\partial x^j} + \Gamma_{ij}^m \Gamma_{mk}^l - \Gamma_{ik}^m \Gamma_{mj}^l .$$

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References

- [1] G. S. Asanov, *Gauge-Covariant Stationary Curves on Finslerian and Jet Fibrations and Gauge Extension of Lorentz Force*, Tensor N. S., No. **50** (1991), 122-137.
- [2] G. S. Asanov, *Jet Extension of Finslerian Gauge Approach*, Fortschritte der Physik **38**, No. **8** (1990), 571-610.
- [3] L. A. Cordero, C. T. J. Dodson, M. de Léon, *Differential Geometry of Frame Bundles*, Kluwer Academic Publishers, 1989.
- [4] J. Eells, L. Lemaire, *A Report on Harmonic Maps*, Bulletin of London Mathematical Society **10** (1978), 1-68.
- [5] G. Giachetta, L. Mangiarotti, G. Sardanashvily, *Covariant Hamiltonian Field Theory*, <http://xxx.lanl.gov/hep-th/9904062>, (1999).
- [6] M. J. Gotay, J. Isenberg, J. E. Marsden, *Momentum Maps and the Hamiltonian Structure of Classical Relativistic Fields*, <http://xxx.lanl.gov/hep/9801019>, (1998).
- [7] D. D. Holm, J. E. Marsden, T. S. Rațiu, *The Euler-Poincaré Equations and Semidirect Products with Applications to Continuum Theories*, <http://xxx.lanl.gov/chao-dyn/9801015>, (1998).
- [8] N. Kamran, P. J. Olver, *Le Problème d'Equivalence à une Divergence prés dans le Calcul des Variations des Intégrales Multiple*, C. R. Acad. Sci. Paris, t. **308**, Série I, (1989), 249-252.

- [9] P. W. Michor, T. S. Rațiu, *On the Geometry of the Virasoro-Bott Group*, Helderman-Verlag, No. **8** (1998), 293-309.
- [10] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, 1994.
- [11] R. Miron, M. S. Kirkovits, M. Anastasiei, *A Geometrical Model for Variational Problems of Multiple Integrals*, Proc. of Conf. of Diff.Geom. and Appl., June 26-July 3, (1988), Dubrovnik, Yugoslavia.
- [12] M. Neagu, *From Euler-Lagrange Equations to Canonical Nonlinear Connections*, Archivum Mathematicum (2005), in press.
- [13] M. Neagu, *Generalized Metrical Multi-Time Lagrange Geometry of Physical Fields*, Journals from de Gruyter, Forum Mathematicum, Vol. **15**, No. **1** (2003), 63-92.
- [14] M. Neagu, *Riemann-Lagrange Geometrical Background for Multi-Time Physical Fields*, Balkan Journal of Geometry and Its Applications, Vol. **6**, No. **2** (2001), 49-71.
- [15] M. Neagu, *The Geometry of Relativistic Rheonomic Lagrange Spaces*, Proceedings of Workshop on Diff. Geom., Global Analysis, Lie Algebras, No. **5**, 142-169, (2001); Editor: Prof. Dr. Grigorios Tsagas, Aristotle University of Thessaloniki, Greece.
- [16] M. Neagu, C. Udriște, *Torsion, Curvature and Deflection d -Tensors on $J^1(T, M)$* , Balkan Journal of Geometry and Its Applications, Vol. **6**, No. **1** (2001), 29-45.
- [17] Mircea Neagu, Constantin Udriște and Alexandru Oană, *Multi-Time Sprays and h -Traceless Maps on $J^1(T, M)$* Balkan Journal of Geometry and Its Applications, Vol. **10**, No. **2** (2005), 76-92.
- [18] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, 1986.
- [19] P. J. Olver, *Canonical Elastic Moduli*, Journal of Elasticity **19**, 189-212, (1988).
- [20] D. Saunders, *The Geometry of Jet Bundles*, Cambridge University Press, New-York, London, 1989.
- [21] C. Udriște, *Nonclassical Lagrangian Dynamics and Potential Maps*, Proc. of the Conference on Mathematics in Honour of Prof. Radu Roșca at the Occasion of his Ninetieth Birthday, Katholieke University Brussel, Katholieke University Leuven, Belgium, Dec. 11-16 (1999); <http://xxx.lanl.gov/math.DS/0007060>, (2000).
- [22] C. Udriște, *Solutions of DEs and PDEs as Potential Maps using First Order Lagrangians*, Centennial Vranceanu, Romanian Academy, University of Bucharest, June 30-July 4, (2000); <http://xxx.lanl.gov/math.DS/0007061>, (2000); Balkan Journal of Geometry and Its Applications, 6,1(2001), 93-108.
- [23] C. Udriște, M. Postolache, *Atlas of Magnetic Geometric Dynamics*, Monographs and Textbooks 3, Geometry Balkan Press, Bucharest, 2001.

- [24] C. Udriște, M. Ferrara, D. Opreș, *Economic Geometric Dynamics*, Geometry Balkan Press, Bucharest, 2004.
- [25] C. Udriște, *From integral manifolds and metrics to potential maps*, Atti dell'Accademia Peloritana dei Pericolanti, Classe I di Scienze Fis. Mat. et Nat., 81-82, C1A0401008 (2003-2004), 1-16.
- [26] C. Udriște, *Geodesic motion in a gyroscopic field of forces*, Tensor, N. S., 66, 3 (2005), 215-228.

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Jet riemann-hamilton geometrization for the conformal deformed berwald-moo r. Quartic metric depending on momenta. Alexandru OanE~a, Mircea Neagu. In Mironâ€™s physical field geometrical theory, the classical Riemannian Levi-Civita connec-tion is replaced by Cartan canonical connection. It is well-known that in Finsler-Lagrange-Hamilton geometrical framework there are a lot of important linear distinguished (d-) con-nections (Cartan, Berwald, Chern-Rund or Hashiguchi, for instance). However, the use of the Cartan canonical connection is preferred because it is the single linear d-connection which is a metrical connection like the Levi-Civita connection in the classical Riemannian framework. A Riemann-Lagrange geometrization for metrical Multi-Time Lagrange Spaces. Mircea Neagu and Constantin UdriA,ste. Dedicated to the memory of Radu Rosca (1908-2005). Abstract. In this paper we construct a geometrization on the 1-jet bundle $J^1(T, M)$ for the multi-time quadratic Lagrangian function. Our geometrization includes a nonlinear connection \hat{N} , a generalized Cartan canonical \hat{N} -linear connection $\hat{C}\hat{N}$ together with its d-torsions and d-curvatures, naturally provided by the given multi-time quadratic Lagrangian function L . Mathematics Subject Classification: 53B40, 53C60, 53C80. Key words: metrical multi-time Lagrange spaces, nonlinear connections, Cartan canonical connection, d-torsions, d-curvatures. 1 Metrical multi-time Lagrange spaces. â€¢ Generalized Metrical Multi-Time Lagrange Model for General Relativity and Electromagnetism more. by Mircea Neagu. Download (.pdf).â€¢ Riemann-Lagrange Geometrization for Linear ODEs Systems and Superior Order ODEs more. by Mircea Neagu. ABSTRACT. Our geometrization includes a nonlinear connection N , a generalized Cartan canonical N -linear connection (together with its local d-torsions and d-curvatures), naturally provided by a given quadratic Hamiltonian function depending on polymomenta.