International Symposium on Nonlinear Theory and its Applications
Xi'an, PRC, October 7–11, 2002

A Passive Two-step Runge-Kutta Method for the Wave Digital Simulation of Nonlinear Electrical Networks

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Abstract

Wave digital filter principles can be applied to the numerical solution of many kinds of differential equations. It has been shown that, by merging the wave digital concept with RUNGE-KUTTA methods, algorithms of high accuracy can be found which are passive and hence possess a lot of desirable numerical stability properties. In this paper, we introduce a two-step RUNGE-KUTTA method which shares these properties while being even more accurate than a RUNGE-KUTTA method of comparable computational effort.

1. Introduction

Originally, wave digital filters were introduced as a specific class of digital filters which, due to their inherent *passivity*, possess many advantageous properties like e. g. guaranteed stability even under finite word-length conditions [1]. In later publications (e. g. [2]-[4]), it has been shown that the underlying principles may also be used for the digital simulation of electrical networks containing linear and even nonlinear passive elements.

Consider the circuit in Fig. 1 where a resistive voltage source feeds a parallel connection of a resistance and a capacitance through a diode. The standard procedure

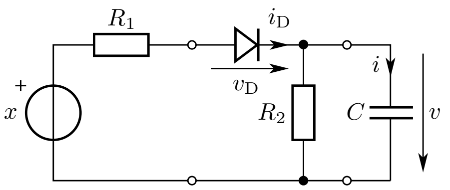

Figure 1: Example for a nonlinear electrical circuit.

to simulate this circuit by means of wave digital structures is as follows: First, the circuit is decomposed into oneports and multiports where to each port a positive port resistance is assigned. Now, so-called (voltage) wave quantities are introduced, i. e., for a port with voltage v_ν, current i_ν, and port resistance R_ν, one defines

$$a_\nu := v_\nu + R_\nu i_\nu \quad \text{and} \quad b_\nu := v_\nu - R_\nu i_\nu. \qquad (1)$$

The equations imposed by both the static network elements as well as KIRCHHOFF laws can immediately be rewritten in dependence on these wave quantities. With suitably chosen port resistances, the resistive voltage source becomes a wave source and the resistance results in a wave sink. The diode described by

$$i_\mathrm{D} = i_\mathrm{D0}\left[\mathrm{e}^{\frac{v_\mathrm{D}}{v_\mathrm{D0}}} - 1\right] \qquad (2)$$

yields a reflected wave quantity $b_\mathrm{D}(a_\mathrm{D})$. The KIRCHHOFF equations describing parallel or series connections are realized by so-called parallel and series adaptors [1].

Finally, the differential relationship describing the linear capacitance,

$$\dot{v}(t) = C^{-1} i(t), \qquad (3)$$

is approximated by the difference equation resulting from an application of a suitably chosen linear multistep (integration) method. In order to characterize the method used, it is common to consider the corresponding difference equation under steady-state condition for a complex frequency p. More precisely, with step size $T > 0$ and $t_k = t_0 + kT$, voltage and current are assumed to be of the form $i(t_k) = I\mathrm{e}^{pt_k}$ and $v(t_k) = V\mathrm{e}^{pt_k}$, respectively, where the complex amplitudes are related by $V = RZI$ with $R = T/[2C]$. Here, we call Z the (normalized) *characteristic impedance*. The relationship between the complex amplitudes of incident and reflected wave reads $B = SA$ where $S = (Z - R)/(Z + R)$ denotes the *scattering function* with respect to a port resistance R.

With these principles, the wave digital elements may be portwise interconnected to obtain the standard wave digital structure which corresponds to the circuit of Fig. 1 and looks as shown in Fig. 2. The static part of the circuit imposes an implicit equation of the form

$$C^{-1}i(t_k) = f(v(t_k), x(t_k)) \tag{4}$$

which is solved by the dynamic-free part of the wave digital structure in the explicit form

$$a(t_k) = g(b(t_k), x(t_k)). \tag{5}$$

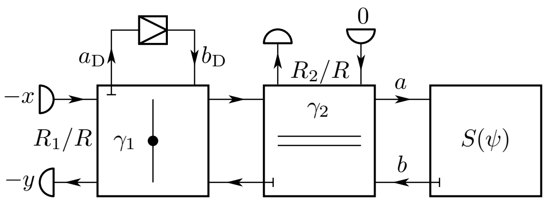

Figure 2: Wave digital structure corresponding to the non-linear electrical circuit shown in Fig. 1.

Now, the driven electrical circuit considered in this paper obviously is a passive one. If the resulting digital simulation algorithm is to possess the same property for arbitrary step sizes T, one is restricted to linear multistep methods where the corresponding characteristic impedance, written in dependence on the complex equivalent frequency $\psi := \tanh(pT/2) = [e^{pT} - 1]/[e^{pT} + 1]$, is a *positive function*, i. e., where $\mathrm{Re}\, Z(\psi) \geq 0$ holds for all ψ with non-negative real part. This requirement limits the approximation accuracy attainable by passive linear multistep methods. It is known that the trapezoidal rule with $Z(\psi) = 1/\psi$ is optimal in this context.

2. Wave Digital Simulation with a 2-stage Passive Runge-Kutta Method

The application of a Runge-Kutta method [6] with two stages to the circuit of Fig. 1 yields the equations

$$\boldsymbol{v}(t_k) = \boldsymbol{e} v(t_k) + 2R\boldsymbol{D}\boldsymbol{i}(t_k) \tag{6}$$

$$v(t_{k+1}) = v(t_k) + 2R\boldsymbol{g}^{\mathrm{T}}\boldsymbol{i}(t_k) \tag{7}$$

with $\boldsymbol{e} = [1,\, 1]^{\mathrm{T}}$, where the parameters of the method are combined in vector $\boldsymbol{g}$ and matrix $\boldsymbol{D}$ of appropriate dimensions. The vectors $\boldsymbol{v}$ and $\boldsymbol{i}$ consist of the stage voltages v_σ and the stage currents i_σ, respectively, which are related by

$$C^{-1}i_\sigma(t_k) = f(v_\sigma(t_k), x(t_k + k_\sigma T)), \quad \sigma = 1, 2 \tag{8}$$

with nodes k_σ and the same function f as given in eq. (4).

For the moment, let us focus on equations (6) and (7) alone. In analogy to the previous section, we may consider these equations for a complex ψ. With this, we define a (normalized) *characteristic impedance matrix* which is given by

$$\boldsymbol{Z}(\psi) = \frac{\boldsymbol{e}\boldsymbol{g}^{\mathrm{T}}}{\psi} + 2\boldsymbol{D} - \boldsymbol{e}\boldsymbol{g}^{\mathrm{T}}. \tag{9}$$

For our purposes, only *passive* Runge-Kutta *methods* are of interest, i. e. methods with a positive characteristic impedance matrix. Clearly, $\boldsymbol{Z}(\psi) + \boldsymbol{Z}^*(\psi)$ is positive semi-definite for all $\mathrm{Re}\,\psi \geq 0$ if, with a weighting vector of the form $\boldsymbol{g}^{\mathrm{T}} = r\boldsymbol{e}^{\mathrm{T}}$, the characteristic impedance matrix is written in the form

$$\boldsymbol{Z}(\psi) = \frac{r}{\psi}\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}} + \boldsymbol{Z}'_{\mathrm{s}} + \boldsymbol{Z}'_{\mathrm{a}} \tag{10}$$

where r is positive, $\boldsymbol{Z}'_{\mathrm{s}}$ is real symmetric positive semi-definite, and $\boldsymbol{Z}'_{\mathrm{a}}$ is real skew-symmetric [7]. Moreover, the method is required to be *diagonally implicit*, i. e., the matrix $\boldsymbol{D} = \boldsymbol{Z}(1)/2$ must be lower triangular. The maximum attainable consistency order this way is three [8] and the corresponding method is specified by

$$\boldsymbol{Z}'_{\mathrm{s}} = \begin{bmatrix} r' & -r' \\ -r' & r' \end{bmatrix} \text{ and } \boldsymbol{Z}'_{\mathrm{a}} = \begin{bmatrix} 0 & r' - r \\ r - r' & 0 \end{bmatrix} \tag{11}$$

with

$$r = \frac{1}{2} \quad \text{and} \quad r' = \frac{1}{2} + \frac{1}{\sqrt{3}} \tag{12}$$

as well as $\boldsymbol{k} = [k_1, k_2]^{\mathrm{T}} = \boldsymbol{D}\boldsymbol{e}$.

Now, we decompose $\boldsymbol{Z}(\psi)$ into the sum

$$\boldsymbol{Z}(\psi) = \boldsymbol{Z}_{\mathrm{p}}(\psi) + \boldsymbol{Z}_{\mathrm{q}}. \tag{13}$$

Again, we require both $\boldsymbol{Z}_{\mathrm{p}}(1)$ and $\boldsymbol{Z}_{\mathrm{q}}$ to be of lower triangular form. The two positive matrices become

$$\boldsymbol{Z}_{\mathrm{p}}(\psi) = \begin{bmatrix} \dfrac{r}{\psi} & \dfrac{r}{\psi} - r \\ \dfrac{r}{\psi} + r & \dfrac{r}{\psi} \end{bmatrix} \tag{14}$$

and

$$\boldsymbol{Z}_{\mathrm{q}} = \begin{bmatrix} r' & 0 \\ -2r' & r' \end{bmatrix}. \tag{15}$$

Next, we set the normalized port resistances for the 2-ports equal to the diagonal entries of $\boldsymbol{Z}_{\mathrm{p}}(1)$ and $\boldsymbol{Z}_{\mathrm{q}}$, respectively. Then, the corresponding scattering matrices are given by

$$\boldsymbol{S}_{\mathrm{p}}(\psi) = \begin{bmatrix} 0 & \frac{1-\psi}{1+\psi} \\ 1 & 0 \end{bmatrix} \text{ and } \boldsymbol{S}_{\mathrm{q}} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}. \tag{16}$$

From these results, wave digital realizations of the two 2-ports are immediately apparent. Due to eq. (13), they

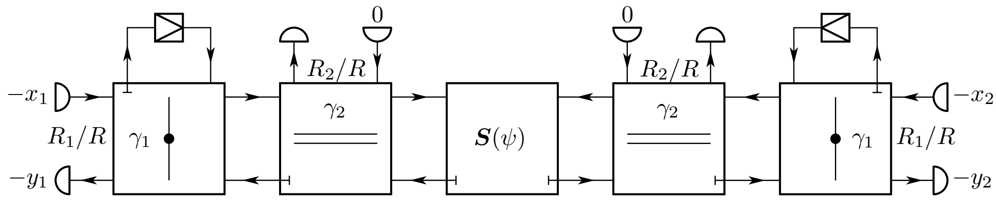

Figure 4: Wave digital structure corresponding to the non-linear electrical circuit shown in Fig. 1 for a 2-stage integration method. The input signals are given by $x_\sigma(t_k) = x(t_k + k_\sigma T)$.

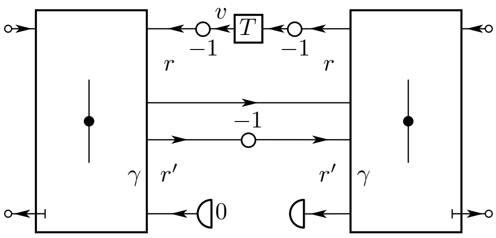

Figure 3: Wave digital structure for the passive 2-stage Runge-Kutta method.

are to be connected in series, which results in the wave digital structure shown in Fig 3.

The nonlinear implicit equations (8) can be solved explicitly for the wave quantities in the form

$$a_\sigma(t_k) = g(b_\sigma(t_k), x(t_k + k_\sigma T)). \qquad (17)$$

With the arrangement shown in Fig. 4, this is done by implementing two identical dynamic-free wave digital structures each of which possesses the same topology as the structure of Fig. 2 and which are connected via the block in the center denoting the 2-port of Fig. 3.

3. A Passive Two-step Runge-Kutta Method

Recently, a general class of two-step Runge-Kutta methods (TSRK methods) has been introduced [9]. In connection with wave digital simulation, we propose to consider the special case where, compared with the equations (6) to (8) of the 2-stage Runge-Kutta method, only (6) is extended according to

$$\boldsymbol{v}(t_k) = \boldsymbol{e}v(t_k) + 2R\boldsymbol{D}\boldsymbol{i}(t_k) + 2R\bar{\boldsymbol{D}}\boldsymbol{i}(t_{k-1}). \qquad (18)$$

Herein, $\bar{\boldsymbol{D}}$ determines the influence of the previous step.

For passivity, the characteristic impedance matrix

$$\boldsymbol{Z}(\psi) = \frac{r}{\psi}\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}} + 2\boldsymbol{D} - r\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}} + 2\frac{1-\psi}{1+\psi}\bar{\boldsymbol{D}} \qquad (19)$$

has to be positive. In the following, we assume $\boldsymbol{D}$, $\boldsymbol{g}$, and $\boldsymbol{k}$ to be the same as for the passive Runge-Kutta method presented above. This means that, after the extraction of the pole at zero, the remaining impedance $\boldsymbol{Z}'(\psi) = \boldsymbol{Z}(\psi) - r\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}/\psi$ can be written in the form

$$\boldsymbol{Z}'(\psi) = \frac{1}{1+\psi}[\boldsymbol{Z}'_{\mathrm{s}} + 2\bar{\boldsymbol{D}}] + \frac{\psi}{1+\psi}[\boldsymbol{Z}'_{\mathrm{s}} - 2\bar{\boldsymbol{D}}] + \boldsymbol{Z}'_{\mathrm{a}} \qquad (20)$$

with $\boldsymbol{Z}'_{\mathrm{s}}$ and $\boldsymbol{Z}'_{\mathrm{a}}$ as in (11). Since $\boldsymbol{Z}'_{\mathrm{s}}$ is positive semidefinite, $\boldsymbol{Z}'(\psi)$ clearly is a positive matrix, if $\bar{\boldsymbol{D}}$ is chosen to be $\bar{\boldsymbol{D}} = -\boldsymbol{Z}'_{\mathrm{s}}/2$. This choice yields a method with consistency order four [10] and will be used from now on.

As before, we decompose the characteristic impedance matrix into two matrices $\boldsymbol{Z}_{\mathrm{p}}(\psi)$ and $\boldsymbol{Z}_{\mathrm{q}}(\psi)$, which are of lower triangular form for $\psi = 1$. This again yields (14) for $\boldsymbol{Z}_{\mathrm{p}}(\psi)$ while $\boldsymbol{Z}_{\mathrm{q}}(\psi)$ now becomes

$$\boldsymbol{Z}_{\mathrm{q}}(\psi) = \begin{bmatrix} \dfrac{2r'}{1+\psi} & \dfrac{r'(1-\psi)}{1+\psi} \\ \dfrac{-r'(1+3\psi)}{1+\psi} & \dfrac{2r'}{1+\psi} \end{bmatrix}. \qquad (21)$$

With the same port resistances as above, the scattering matrix $\boldsymbol{S}_{\mathrm{p}}(\psi)$ is still given by (16) while $\boldsymbol{S}_{\mathrm{q}}(\psi)$ becomes

$$\boldsymbol{S}_{\mathrm{q}}(\psi) = \begin{bmatrix} 0 & \dfrac{(1-\psi)}{1+3\psi} \\ -1 & 0 \end{bmatrix}. \qquad (22)$$

The resulting wave digital structure is shown in Fig. 5.

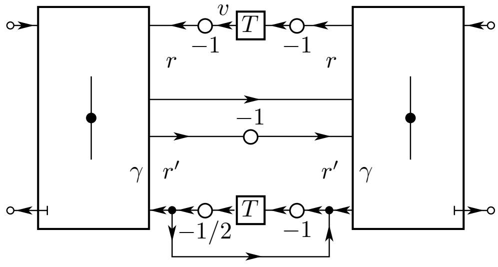

Figure 5: Wave digital structure for the passive 2-stage TSRK method.

Note that there is only a slight change in the wave digital structure when compared with the Runge-Kutta method (Fig. 3) and the computational effort for both methods is roughly the same. Furthermore, in the starting procedure necessary for the TSRK method, the initial value of the additional delay may just be set to zero. In this case, the wave digital structures of Fig. 5 and Fig. 3 coincide which means that in fact the starting procedure is carried out by the RUNGE-KUTTA method.

4. Simulation Results

In the following, wave digital simulation results for the circuit of Fig. 1 will be given for both passive 2-stage methods presented in this paper.

The voltage v_{D} across the diode is set to be of sinusoidal form, whereby the (FOURIER coefficients of the) exact solution for the voltage across the capacitance can be obtained. The input signal, x, for the simulation is then chosen accordingly. Thus, the error induced by the numerical integration can be computed. The network parameters were chosen to be $R_1 = 10\,\Omega$, $R_2 = 1\,\mathrm{k}\Omega$, $C = 1\,\mu\mathrm{F}$, $v_{\mathrm{D0}} = 30\,\mathrm{mV}$, and $i_{\mathrm{D0}} = 10\,\mathrm{pA}$. Amplitude and frequency of v_{D} were 0.65 V and 440 Hz, respectively.

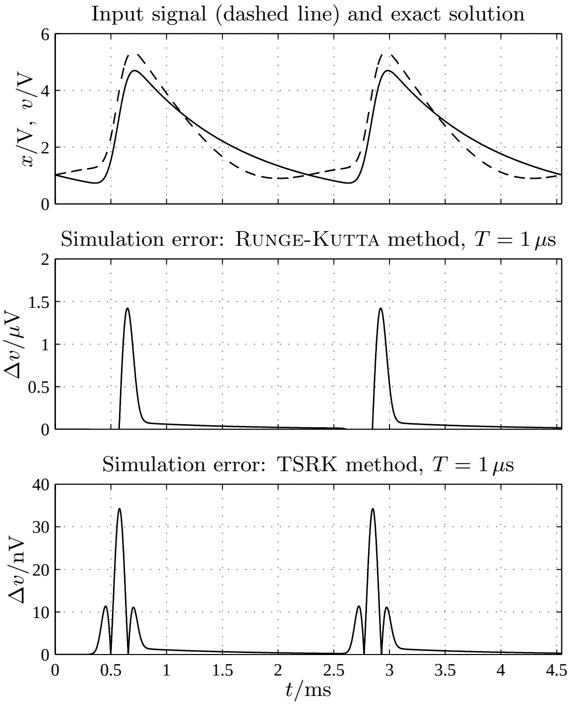

Figure 6: Simulation of the electrical circuit given in Fig. 1: Excitation, exact solution, and simulation errors for the RUNGE-KUTTA and the TSRK method.

From Fig. 6, it becomes obvious that the usage of the passive TSRK method with higher consistency order does indeed increase the approximation accuracy.

5. Summary

In this paper, a passive two-step RUNGE-KUTTA method for the wave digital simulation of electrical circuits has been presented. Simulation results have confirmed that this method is more accurate than a passive RUNGE-KUTTA method of comparable computational effort.

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Approach: The Runge-Kutta method finds an approximate value of y for a given x. Only first-order ordinary differential equations can be solved by using the Runge Kutta 2nd order method. Below is the formula used to compute next value yn+1 from previous value yn. Therefore: yn+1 = value of y at (x = n + 1) yn = value of y at (x = n) where 0 â‰¤ n â‰¤ (x - x0)/h h is step height xn+1 = x0 + h. The essential formula to compute the value of y(n+1): The formula basically computes the next value yn+1 using current yn plus the weighted average of two increments: K1 is the increment based on the slope at the beginning of the interval, using y. K2 is the increment based on the slope at the midpoint of the interval, using (y + h*K1/2). Wave digital simulation of nonlinear electrical networks by means of passive Runge-Kutta methods. Proceedings of IEEE International Symposium on Circuits, Systems, III , 469â€"472, Sydney, Australia. Fries, M. (1994a, April).Â X. Wang (1998) The wave digital pde simulation method: Some applications and machine architecture considerations University of Notre Dame IN, USA. Google Scholar. X. Wang S. Bass (1997) The wave digital method and its use in a PIM chip array University of Notre Dame IN, USA. Google Scholar. C.Q. Xu (1996) Accommodating lumped linear boundary conditions in the wave digital simulations of PDE systems University of Notre Dame IN, USA. Google Scholar. I have two algorithms for a numerical differential equation problem, one called Euler's method and one called a second-order Runge Kutta(RK2) . Essentially Euler's method and RK2 approximate a solution to differential equations. The only difference is that they use different formulas (Euler's uses a first order derivative of Taylor's series whilst the RK2 is a second order derivative of a Taylor series).Â I have provided the code that I used to create this below followed by a numeric example of the second-order Runge Kutta method that is working numerically. I am interested in showing that the convergence is quicker with the RK2 than Euler's method. import matplotlib.pyplot as plt import numpy as np from math import exp # exponential function. 5 Relation to water wave simulation. 6 Numerical examples. arXiv:1712.04881v1 [math.NA] 13 Dec 2017.Â The numerical methods can roughly be divided into two classes depending on whether the conformal mapping is used or not. In [3], the waterwave problems were solved by an integral formulation and its discretization (see Section 5 for more information). However, the convergence was proved with time variable being kept continuous.Â We derive in the following the stability conditions of general Runge-Kutta methods when applied to this system (3.2), including explicit and implicit schemes. In particular, we aim to investigate optimal stability conditions when explicit Runge-Kutta methods are used.