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結合的グラスマン多様体
Lagrangian submanifolds of S^6 and the associative Grassmann manifold

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It is well-known that a six-dimensional sphere S^6 admits an almost complex structure defined by its natural inclusion in the space $\mathrm{Im}\mathbb{O}$ of imaginary octonions. This almost complex structure is not integrable but is nearly Kähler with respect to the induced Riemannian metric from the inner product in $\mathrm{Im}\mathbb{O}$.

An oriented three-dimensional subspace of $\mathrm{Im}\mathbb{O}$ is said to be associative if it is a canonically oriented imaginary part of some quaternion subalgebra of $\mathbb{O}$. The set of all associative subspaces is called the associative Grassmann manifold, which is denoted by $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. Then it is known that $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is an eight-dimensional compact symmetric quaternionic Kähler manifold.

We focus on Lagrangian submanifolds of S^6 and study the relationship of such submanifolds with the geometry of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$.
This is a joint work with K.Enoyoshi ([3]).

§1 The algebra of octonions and the Lie group G_2

In this section, we recall fundamental properties of octonions following R. Harvey and H. B. Lawson [5]. Let $\mathbb{H} = \{x1+yi+zj+wk \mid x,y,z,w \in \mathbb{R}\} \cong \mathbb{R}^4$ ($i^2=j^2=k^2=-1, ij=-ji=k$) be the algebra of quaternions and $Sp(1)$ the group of unit quaternions. We denote the subspace of imaginary quaternions by $\mathrm{Im}\mathbb{H}$. The algebra $\mathbb{O}$ of octonions is a normed algebra whose the multiplication is given by

$$(a+b\varepsilon)(c+d\varepsilon) = (ac-\bar{d}b) + (da+b\bar{c})\varepsilon \qquad a,b,c,d \in \mathbb{H}$$

([5]). Here $\bar{a}$ is the quaternionic conjugation for $a \in \mathbb{H}$. The algebra $\mathbb{O}$ is neither commutative nor associative. Let $\mathrm{Im}\mathbb{O} = \mathrm{Im}\mathbb{H} \oplus \mathbb{H}\varepsilon$ be the subspace of all imaginary parts of octonions, which is identified with seven-dimensional Euclidian space $\mathbb{R}^7$.

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We define the alternating trilinear form φ on $\mathrm{Im}\mathbb{O}$ by

$$\varphi(x, y, z) = \langle x, yz \rangle.$$

The 3-form φ is called the *associative calibration* on $\mathrm{Im}\mathbb{O}$ ([5] p.113 Definition 1.5). The Lie group G_2 is defined by

$$G_2 = \mathrm{Aut}(\mathbb{O}) = \{g \in GL(\mathbb{O}) |\ g(xy) = g(x)g(y),\ \text{for any } x, y \in \mathbb{O}\}.$$

It is well-known that the Lie group G_2 is 14-dimensional and simple ([5]). Every automorphism of $\mathbb{O}$ fixes the subspace $\mathbb{R} \cdot 1 \subset \mathbb{O}$ and leaves the subspace $\mathrm{Im}\mathbb{O}$ invariant. We also have the facts that G_2 is a subgroup of $SO(\mathrm{Im}\mathbb{O}) \cong SO(7)$ and that the following holds:

$$G_2 = \{g \in O(7) |\ g^*\varphi = \varphi\}.$$

For the pair of unit quaternions $(q_1, q_2) \in Sp(1) \times Sp(1)$ and $a + b\varepsilon \in \mathbb{O} = \mathbb{H} \oplus \mathbb{H}\varepsilon$, we set

$$\rho(q_1, q_2)(a + b\varepsilon) = q_1 a q_1^{-1} + (q_2 b q_1^{-1})\varepsilon.$$

Then we see that $\rho(q_1, q_2)$ belongs to G_2 and the kernel of ρ is $\{\pm(1, 1)\}$. Hence ρ is an action of $Sp(1) \times Sp(1)/\{\pm(1, 1)\} \cong SO(4)$ on $\mathbb{O}$.

§2 The associative Grassmann manifold $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ and its tangent space

We denote by $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ the Grassmann manifold of all oriented three-dimensional subspaces in $\mathrm{Im}\mathbb{O}$ with $\dim \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O}) = 12$. $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ is isomorphic to the Riemannian symmetric space $SO(7)/SO(3) \times SO(4)$. If $\zeta \in \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ is the canonically oriented imaginary part of some quaternion subalgebra of $\mathbb{O}$, then ζ is said to be an *associative subspace*. The set of all associative subspaces is called the *associative Grassmann manifold* denoted by $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. Then it is known that $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is an eight-dimensional compact symmetric quaternionic Kähler manifold which is described as $G_2/SO(4)$ (cf. [5],[11]). Moreover we see that $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is a totally geodesic submanifold of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$.

The associative calibration φ can be viewed as a function on $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$. The following has been shown by Harvey and Lawson;

Proposition 2.1 ([5]) For $\zeta \in \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$, $\varphi(\zeta) \leq 1$ with the equality if and only if ζ is associative.

Then we define level sets $\tilde{M}(t)$ of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ for $-1 \leq t \leq 1$ by

$$\tilde{M}(t) = \{\zeta \in \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O}) \mid \varphi(\zeta) = t\}.$$

The Lie group G_2 acts transitively on each $\tilde{M}(t)$ $(-1 \leq t \leq 1)$. The level set $\tilde{M}(1)$ coincides with the associative Grassmann manifold $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. Reversing the orientation of subspaces in $\tilde{M}(-1)$, we see that $\tilde{M}(-1)$ is isometric to $\tilde{M}(1)$. For $-1 < t < 1$, $\tilde{M}(t)$ is diffeomorphic to $G_2/SO(3)$.

We will describe the tangent space of the associative Grassmann manifold $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ following F. Nakata ([9]). By the standard argument using the theory of vector bundles for the differential geometry of Grassmann manifolds, the tangent space of the Grassmann manifold at $\xi \in \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ is identified with the space $\mathrm{Hom}(\xi, \xi^\perp)$ of linear homomorphisms of ξ to $\xi^\perp$, where $\xi^\perp$ denotes the orthogonal complement of ξ in $\mathrm{Im}\mathbb{O}$. The tangent space of the associative Grassmann manifold $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is described as follows:

$$T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}) \simeq \{\gamma \in \mathrm{Hom}(\xi, \xi^\perp) \mid \gamma(e_1)e_1 + \gamma(e_2)e_2 + \gamma(e_3)e_3 = 0\},$$

where $\{e_1, e_2, e_3\}$ denotes an orthonormal basis of ξ. This description is due to Nakata ([9]). We denote the right hand side in the above equation by $\mathrm{Hom}_{ass}(\xi, \xi^\perp)$.

§3 The nearly Kähler structure and submanifolds of S^6

Let S^6 be the unit sphere in $\mathrm{Im}\mathbb{O}$ centered at the origin. Then S^6 has an almost complex structure J at $p \in S^6$ defined by

$$J_p x = px, \quad x \in T_p S^6.$$

It is known that this almost complex structure J is not integrable. The Riemannian metric on S^6 is induced from the inner product $\langle , \rangle$ on $\mathrm{Im}\mathbb{O} \cong \mathbb{R}^7$. The induced metric $\langle , \rangle$ is Hermitian with respect to J. Then we define the Kähler form ω of S^6 at $p \in S^6$ by

$$\omega_p(x, y) = \langle J_p x, y \rangle, \quad x, y \in T_p S^6.$$

Then $(S^6, J, \langle , \rangle)$ is an almost Hermitian manifold. Moreover it satisfies $(\tilde{\nabla}_X J)X = 0$ for all vector fields X on S^6 , where $\tilde{\nabla}$ denotes the Riemannian connection. An almost Hermitian manifold with this property is called a *nearly Kähler manifold*. We have the fact that the group of automorphisms of $(S^6, J, \langle , \rangle)$ is isomorphic to G_2.

The nearly Kähler six-sphere has two typical classes of submanifolds: namely the class of almost complex submanifolds and that of Lagrangian submanifolds or three-dimensional totally real submanifolds. A. Gray ([4]) showed that there do not exist four-dimensional almost complex submanifolds in S^6. R. L. Bryant([1]) proved that every compact Riemann surface can be realized as an almost complex curve in S^6. Many researchers have studied two-dimensional almost complex submanifolds or almost complex curves and obtained fruitful results.

We recall Lagrangian submanifolds of six-sphere. We define a trilinear form ϕ and a complex-valued form Ω on S^6 using the associative calibration φ as follows:

$$\begin{aligned} \phi(x, y, z) &= \varphi(x, y, J_p z) \qquad x, y, z \in T_p S^6 \\ \Omega(x, y, z) &= \phi(x, y, z) + \sqrt{-1}\,\varphi(x, y, z) \end{aligned}$$

Then Ω is a $(3,0)$-form with respect to J. We have the following interesting relations between the Kähler form and the associative calibration.

Proposition 3.1 (cf. K.Mashimo [8]) For $p \in S^6$,
(1) $\omega_p = p \rfloor \varphi$
(2) $d\omega = 3\varphi|_{S^6}$
(3) $d\phi = 4\omega \wedge \omega$

We state the special Lagrangian geometry of S^6 with respect to Ω. A three-dimensional subspace ζ of T_pS^6 is called a *Lagrangian subspace* if it holds that $J_px \perp \zeta$ for all $x \in \zeta$. The condition is equivalent to $\omega|_\zeta = 0$. We call a three-dimensional submanifold M of S^6 a *Lagrangian submanifold* if the tangent space T_pM is a Lagrangian subspace at each point $p \in M$. The condition implies that ω restricted to M vanishes. Many researchers refer to Lagrangian submanifolds in S^6 as three-dimensional totally real submanifolds of S^6. The following remarkable theorem by N.Ejiri is known:

Theorem 3.2([2]) Any Lagrangian submanifold of S^6 is orientable and minimal.

We call an oriented Lagrangian subspace ζ a *special Lagrangian subspace* with respect to Ω if $\Omega(v_1, v_2, v_3) = 1$ for a positively oriented orthonormal basis $\{v_1, v_2, v_3\}$ of ζ. An oriented Lagrangian submanifold M is called a *special Lagrangian submanifold* if the tangent space T_pM is a special Lagrangian subspace at each point $p \in M$. From the view point of calibrated geometry, Mashimo showed the following:

Theorem 3.3([8]) If a three-dimensional oriented submanifold M of S^6 is Lagrangian, then it is a special Lagrangian submanifold, if necessary, we reverse its orientation.

We review basic facts on special Lagrangian subspaces of the tangent space of S^6.

Lemma 3.4 Let ζ be a three-dimensional oriented subspace of T_pS^6. Then either ζ or $-\zeta$ is special Lagrangian if and only if ζ is a Lagrangian subspace and $\varphi(\zeta) = 0$.

Lemma 3.5 Let ζ be a three-dimensional oriented subspace of T_pS^6. If ζ is special Lagrangian, $-J_p\zeta$ is associative. Conversely, if ζ is associative, $J_p\zeta$ is special Lagrangian.

§4 The two double fibrations

Nakata ([9]) showed the following double fibration motivated to construct a theory of Penrose type twistor correspondence for the geometries of S^6 and $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. Let $\widetilde{\mathrm{Fl}}_{1,ass}(\mathrm{Im}\mathbb{O})$ be a flag manifold defined by

$$\widetilde{\mathrm{Fl}}_{1,ass}(\mathrm{Im}\mathbb{O}) = \{(p,\xi) \in S^6 \times \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}) \mid p \in \xi \}.$$

We define maps $\bar{\omega}$ and π_- of $\widetilde{\mathrm{Fl}}_{1,ass}(\mathrm{Im}\mathbb{O})$ onto S^6 and $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ by the projections to the first factor and the second factor, respectively.

(4.1)
$$\begin{array}{ccc} & \widetilde{\mathrm{Fl}}_{1,ass}(\mathrm{Im}\mathbb{O}) & \\ {\scriptstyle \bar{\omega}} \swarrow & & \searrow {\scriptstyle \pi_-} \\ S^6 & & \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}) \end{array}$$

These fibrations are G_2-equivariant. He proved the following interesting result:

Theorem 4.1([9]) In the double fibration (4.1), for each $p \in S^6$, $\pi_-(\bar{\omega}^{-1}(p))$ is a four-dimensional totally geodesic and quaternionic submanifold of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ which is isomorphic to $\mathbb{C}P^2$ and for each $\xi \in \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$, $\bar{\omega}(\pi_-^{-1}(\xi))$ is a two-dimensional sphere S^2 which is totally geodesic and almost complex in S^6.

We consider another double fibration to associate the geometry of the associative Grassmann manifold with Lagrangian submanifolds of S^6. As a preparation of constructing the fibration, we recall the triple cross product:

$$x \times y \times z = \frac{1}{2}\left(x(\bar{y}z) - z(\bar{y}x)\right) \qquad \text{for } x, y, z \in \mathbb{O},$$

where $\bar{y}$ is the octonion conjugation for $y \in \mathbb{O}$. It is known that $\langle x, y \times z \times w\rangle$ are alternating in $x, y, z, w \in \mathbb{O}$ and that the real part of $x \times y \times z = \varphi(x, y, z)$ for $x, y, z \in \mathrm{Im}\mathbb{O}$. We recall the level set $\tilde{M}(0)$ defined in Section 2 :

$$\tilde{M}(0) = \{\zeta \in \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O}) \mid \varphi(\zeta) = 0\}.$$

Let $\{v_1, v_2, v_3\}$ be a positively oriented orthonormal basis of $\zeta \in \tilde{M}(0)$ and put $p = -v_1 \times v_2 \times v_3 = v_1(v_2 v_3)$. Then since the real part of $v_1 \times v_2 \times v_3$ is equal to $\varphi(\zeta)$, we have $p \in \mathrm{Im}\mathbb{O}$. Moreover, it holds that $|p| = 1$ and p is orthogonal to ζ. Then ζ is a subspace of T_pS^6. Moreover ζ is a special Lagrangian subspace, hence by Lemma 3.5, $-J_p\zeta$ is associative. By these, we can define another double fibration:

(4.2)
$$\begin{array}{ccc} & \tilde{M}(0) & \\ {\scriptstyle \chi} \swarrow & & \searrow {\scriptstyle \pi} \\ S^6 & & \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}). \end{array}$$

$$\chi(\zeta) = -v_1 \times v_2 \times v_3 = p, \quad \pi(\zeta) = -J_p\zeta$$

These fibrations are also G_2-equivariant. For this double fibration, the similar result to Nakata' one holds:

Theorem 4.2 In the double fibration (4.2), for each $p \in S^6$, $\pi(\chi^{-1}(p))$ is a five-dimensional totally geodesic submanifold of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ which is isomorphic to $SU(3)/SO(3)$ and for each $\xi \in \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$, $\chi(\pi^{-1}(\xi))$ is a three-dimensional sphere S^3 which is totally geodesic and special Lagrangian in S^6.

Remark. S.Klein gives the classification of the totally geodesic submanifolds of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ in [6]. In the following table, we list the maximal totally geodesic submanifolds of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$.

maximal tot. geod.	dim.
$SU(3)/SO(3)$	5
$\mathbb{C}P^2$	4
$S^2_{r=1} \times S^2_{r=\frac{1}{\sqrt{3}}}/\mathbb{Z}_2$	4
$S^2_{r=\frac{2\sqrt{21}}{3}}$	2

Theorems 4.1 and 4.2 give the geometric realization of totally geodesic submanifolds $\mathbb{C}P^2$ and $SU(3)/SO(3)$.

It is important and useful for the submanifolds geometry of Riemannian symmetric spaces to characterize the tangent spaces of totally geodesic submanifolds, so called, curvature-invariant subspaces, or Lie triple systems. For our cases, we have the following: For $p \in S^6$, we put

$$\mathfrak{S}_p = \pi_-(\bar{\omega}^{-1}(p)) = \mathbb{C}P^2, \quad \mathcal{L}_p = \pi(\chi^{-1}(p)) = SU(3)/SO(3).$$

Then for $\xi \in \mathfrak{S}_p$ and $\xi \in \mathcal{L}_p$, we have the following:

$$\begin{aligned} T_\xi \mathfrak{S}_p &= \{ \gamma \in T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}) \mid \gamma(p) = 0 \}, \\ T_\xi \mathcal{L}_p &= \{ \gamma \in T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}) \mid \langle \gamma(v), p \rangle = 0 \text{ for any } v \in \xi \}, \end{aligned}$$

under the identification of $T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ with $\mathrm{Hom}_{ass}(\xi, \xi^\perp)$. Here we note that in the case of $\mathbb{C}P^2$, $p \in \xi$ and in the case of $SU(3)/SO(3)$, $p \in \xi^\perp$. They give the geometric description of the Lie triple systems. Here we note that $\mathfrak{S}_p = \mathfrak{S}_{-p}$, $\mathcal{L}_p = \mathcal{L}_{-p}$.

For $\xi \in \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ and a one-dimensional subspace l of $\xi^\perp$, we put

$$\mathfrak{m}_l = \{ \gamma \in T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}) \mid \langle \gamma(v), l \rangle = 0 \text{ for any } v \in \xi \}.$$

Then $\mathfrak{m}_l$ is a Lie triple system and coincides with the tangent space of $\mathcal{L}_p$ at ξ, where $p \in l \cap S^6$. A five-dimensional subspace $\mathfrak{m}$ of $T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is called a *SL-type subspace* if there exists a one-dimensional subspace l of $\xi^\perp$ such that $\mathfrak{m} = \mathfrak{m}_l$. For later use, we prepare the following Lemma:

Lemma 4.3 Let $\mathfrak{p}$ be a three-dimensional subspace of $T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. If there exists a SL-type subspace $\mathfrak{m}$ such that $\mathfrak{p} \subset \mathfrak{m}$, it is unique.

§5 The Gauss maps of submanifolds of S^6 to $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$

We show remarkable relations between Lagrangian submanifolds of S^6 and the geometry of the associative Grassmann manifold.

Let $f : M \to S^6$ be a Lagrangian immersion of a three-dimensional manifold M into S^6. Then by Theorem 3.3, f is special Lagrangian. By Lemma 3.5, $-J_{f(p)} df(T_pM)$ is associative. We obtain a kind of Gauss map $\nu : M \to \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ defined by $\nu(p) = -J_{f(p)} df(T_pM)$.

Theorem 5.1 If $f : M \to S^6$ is a Lagrangian immersion, then the Gauss map $\nu : M \to \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ associated to f is harmonic.

It is a similar result as a famous formula by E. A. Ruh and J. Vilms ([10]).

Next we consider a reconstruction of Lagrangian immersions from maps to the associative Grassmann manifold. First we define a new class of maps to $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. Let $\bar{\nu}\colon M \to \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ be a smooth map of a three-dimensional manifold M to $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. We call $\bar{\nu}$ an *inclusive map of SL-type* if there exists a SL-type subspace $\mathfrak{m}$ of $T_{\bar{\nu}(p)}\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ at each point $p \in M$ such that $d\bar{\nu}(T_pM) \subset \mathfrak{m}$. Then we have the fact that the Gauss map ν associated to a Lagrangian immersion $f\colon M \to S^6$ is an inclusive map of SL-type. Conversely, let $\nu\colon M \to \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ be an inclusive immersion of SL-type of a three-dimensional manifold M. Then taking non-trivial double covering $\eta\colon M' \to M$, if necessary, we obtain a map $f\colon M' \to S^6$ which satisfies $f(p) \in (\nu \circ \eta(p))^{\perp}$ and $d(\nu \circ \eta)(T_pM') \subset \mathfrak{m}_{\mathbb{R}f(p)}$ at each $p \in M'$ due to Lemma 4.3.

Proposition 5.2 If f is an immersion, then f is Lagrangian and $\nu \circ \eta$ is the Gauss map associated to f and hence harmonic with respect to the induced Riemannian metric by f.

§6 Examples $\cdots$ Homogeneous Lagrangian submanifolds

Mashimo ([8]) classified compact Lagrangian submanifolds of S^6 which are obtained as orbits of closed subgroups of G_2. That is, it is a totally geodesic spehere or it is congruent to one of four kinds of Lagrangian submanifolds M_i $(i = 1, 2, 3, 4)$. They are orbits of three-dimensional Lie subgroups $SU(2)$ or $SO(3)$. The following is easy to prove:

Proposition 6.1 The Gauss map associated to a totally geodesic and Lagrangian submanifold is constant.

The Gauss maps $\nu_i\colon M_i \to \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ associated to M_i $(i = 1, \dots, 4)$ are equivariant under the corresponding Lie groups. In this note, we explain M_3 and M_4 only. The following description of the action by $SO(3)$ is due to J. D. Lotay ([7]): We identify $\mathrm{Im}\mathbb{O}$ with the space $\mathcal{H}^3(\mathbb{R}^3)$ of homogeneous harmonic cubics on $\mathbb{R}^3$ by the following correspondence:

$$\begin{array}{ll} e_1 \mapsto \frac{\sqrt{10}}{10}x(2x^2 - 3y^2 - 3z^2); & \\ e_2 \mapsto -\sqrt{6}xyz; & e_3 \mapsto \frac{\sqrt{6}}{2}x(y^2 - z^2); \\ e_4 \mapsto -\frac{\sqrt{15}}{10}y(4x^2 - y^2 - z^2); & e_5 \mapsto -\frac{\sqrt{15}}{10}z(4x^2 - y^2 - z^2); \\ e_6 \mapsto \frac{1}{2}y(y^2 - 3z^2); & e_7 \mapsto -\frac{1}{2}z(z^2 - 3y^2). \end{array}$$

Then the standard $SO(3)$ action on $\mathbb{R}^3$ induces an action on $\mathcal{H}^3(\mathbb{R}^3)$. Then by this action $SO(3)$ is a subgroup of G_2. Let M_3 and M_4 be the orbits through e_2 and e_6 of this $SO(3)$-action, respectively. Then $M_3 = SO(3)/A_4$ and $M_4 = SO(3)/S_3$ are Lagrangian submanifolds of S^6. In particular, M_3 is of constant curvature $\frac{1}{16}$.

We compute the differentials $d\nu_i\colon T_xM_i \to T_{\nu_i(x)}\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ of the Gauss maps at $x \in M_i$ $(i = 3, 4)$, respectively, and determine the ranks of $d\nu_i$.

Proposition 6.2 The rank of $d\nu_3$ is equal to 3 and the rank of $d\nu_4$ is equal to 2.

Especially in the case of M_3, the following holds:

Proposition 6.3 The Gauss map $\nu_3 \colon M_3 \to \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is a minimal immersion with respect to $\frac{15}{8}\langle , \rangle$, where $\langle , \rangle$ is the metric on M_3.

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Given a Lagrangian submanifold in a symplectic manifold and a Morse function on the submanifold, we show that there is an isotopic Morse function and a symplectic Lefschetz pencil on the manifold extending the Morse function to the whole manifold. From this construction we dene a sequence of symplectic invariants classifying the isotopy classes of Lagrangian spheres in a symplectic 4-manifold. 1. Introduction.Â It is also worth mentioning the relationship between our result and the work of Seidel, who showed that if two vanishing cycles of a Lefschetz pencil can be joined by a â€œmatching pathâ€ (see Section 8), then the total space of the pencil contains a Lagrangian sphere bered over an arc in CP1; in the case where L âˆ¼= Sn. The notion of graded Lagrangian submanifold serves to fix this ambiguity. We explain the theory in detail and give several applications. RESUME.Â Both the construction and the proof can be generalized to produce Lagrangian n-spheres with the same property for all even n. Here, using the method of graded Lagrangian submanifolds, we will first reprove the result from [30] and its generalization in a considerably simpler way. Then, by a more complicated construction, we produce similar examples of Lagrangian n-spheres for all odd n > 5. The reason why the remaining case n = 3 cannot be settled in the same way is topological, and seems to have nothing to do with Floer theory. A Lagrangian Manifold is defined as a submanifold of a symplectic manifold upon which the restriction of a symplectic form ω is vanishing. I'd like to understand what this property means and how the flow of a Lagrangian submanifold differs from that of an arbitrary submanifold. To summarise: What does the condition of vanishing symplectic for imply on a Lagrangian Manifold? What make Lagrangian submanifold different to other submanifolds in the context of Hamiltonian Mechanics? Thank you in advance for your help. sg.symplectic-geometry lagrangian-submanifolds.