Random Dynamical Systems and Multiplicative Ergodic Theorems Workshop (15w5059)

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1 Overview of the Field

The conference was intended to broadly address recent developments in random dynamical systems. Modern research in random dynamical systems comes principally from two directions: stochastic partial differential equations and autonomous (non-random) dynamical systems. Both strands, which have traditionally worked largely in isolation from each other, were well represented at the workshop. We estimate that few if any of the participants (including the organizers) had previously met more than 50% of the other participants.

Some of the central themes that emerged during the workshop were as follows:

Synchronization: This is when trajectories of random dynamical systems *subjected to the same randomness, but starting from different initial configurations* converge in time to a single (random) solution. This appeared in at least 5 talks over the course of the workshop. Crudely speaking, in order to see synchronization, one needs two ingredients: local contraction (negative Lyapunov exponents) so that nearby points approach each other; along with a global irreducibility condition.

Derandomization: By building the realization of the randomness process into the current state of the system, random dynamical systems can often be represented as deterministic dynamical systems on enlarged state spaces. This is a point of entry for the application of the deterministic dynamical systems theory to these questions.

SPDEs as limits of deterministic dynamical systems: This is in a sense the converse of derandomization. Given a fast-slow system (where one set of variables evolves at a rapid rate and a second set evolves slowly, with mutual influence between the two), under suitable conditions, the influence of the fast variables on the slow variables may be averaged, so that the slow variables (in the limit where the ratio of the rates of evolution diverges to infinity) evolve according to an explicit stochastic partial differential equation.

This draws on a large body of work over several decades by a number of authors studying what can be described as dynamical limit theorems and related work. For these results, one is studying the behaviour of the sequence of partial sums of an 'observable' (a function) evaluated at regular intervals along the orbit of a dynamical system. If the dynamical system is sufficiently hyperbolic and the function is sufficiently smooth, one has a decay of correlation between the summands, so that it may not be too surprising that the distribution of the partial sum up to time N is approximately normal for sufficiently large N. This statement is known as a dynamical central limit theorem.

More refined questions (dynamical invariance principles) study versions of the graph of the interpolated partial sums with suitable scalings of both time and displacement. In this case, under suitable hypotheses, one shows that the scaled partial sum trajectory converges in distribution to a Brownian motion.

Multiplicative ergodic theorems: In the dynamical systems literature, for many years the focus was restricted to dynamical systems where the noise was independent and identically distributed. In practice, this means that one has a family of maps, and at each stage a map is randomly selected and applied to give a new point of the orbit. The independence in the map selection made it possible to describe the evolution of the random dynamical system using *annealed* evolution operators, in which one computes the average distribution of the random system as the state evolves. The independence can be shown to ensure that the distribution at the $(n+1)$st time step is a function of the distribution at the nth time step. This is no longer true if the maps are not applied in an i.i.d. manner.

Since the assumption of i.i.d. random maps is not reasonable in many physical situations (e.g. where a dynamical system is forced by a slowly moving much more massive dynamical system), there was a substantial gap between the theory and desired applications. One way to deal with this issue that has seen a great deal of development in the last 5–10 years is the use of the Oseledets multiplicative ergodic theorem (MET) to understand non-i.i.d.forcing of dynamical systems. An informal statement of the MET is that one has a *base* dynamical system and for each point of this system, one has a matrix. As one iterates the base dynamical system, one takes the product of the matrices along the orbit. This structure (a multiplicative cocycle) occurs widely in dynamical systems. The MET may be loosely seen as an analogue of the Jordan normal form, where the space on which the matrices act ($\mathbb{R}^d$) is decomposed into a direct sum of vector subspaces, each expanding or contracting at a different rate under the iterated matrix products.

Recent generalizations of the MET replace matrices by operators acting on Banach spaces. Here, rather than considering the annealed system, one uses the MET to prove structural properties of the *quenched* evolution operators (that is where one studies one realization of the randomness at a time).

Bifurcations of random dynamical systems: It has been known for a long time that the addition of (additive) noise can destroy bifurcations. A simple mechanism for this is that noise allows a random dynamical system to move from one fixed point to another. This mechanism can be curtailed or prevented if instead of adding unbounded noise, the noise is constrained to lie in a bounded region.

2 Recent Developments and Open Problems

The workshop had an open problem session early in the week, where all participants were invited to present a problem or topic for discussion. There were presentations, as follows.

1. Thomas Kaijser:

   In work from the 1970's and 80's, Kaijser showed that i.i.d. random dynamical systems with the property that the maps are bi-Lipschitz (condition R), and satisfy a minimality condition allowing any initial point to be moved arbitrarily close to a desired target point in a finite number of steps (condition M) that satisfy an additional contraction property (condition B or G) have the property that they satisfy synchronization. As a result, there is a unique invariant measure. Kaijser's question was regarding the necessity of the contraction condition.

2. Ian Melbourne:

   Abstract: This conjecture was presented at the open question session on Monday. A successful resolution would extend the scope of the theory presented in the talk "Stochastic limits for deterministic fast-slow systems" given on Thursday.

Moment estimates Let $T : X \to X$ be a measure preserving transformation on a probability space (X, μ). Let P denote the transfer operator, so $\int Pv\, w\, d\mu = \int v\, w \circ T\, d\mu$ for all $v \in L^1$, $w \in L^\infty$.

Suppose that v is a mean zero L^p observable, $p \geq 2$. If $\{v \circ T^j,\ j \geq 0\}$ is a sequence of martingale differences, then it is standard that

$$\Big\| \sum_{j=0}^{n-1} v \circ T^j \Big\|_p \leq C n^{1/2}, \tag{1}$$

for some constant C (depending on v). More generally, if $v = m + \chi \circ T - \chi$ where $m, \chi \in L^p$ and $\{m \circ T^j,\ j \geq 0\}$ is a sequence of martingale differences, then (1) still holds.

For large classes of dynamical systems, sufficiently regular observables v admit a decomposition of the form

$$v = m + \chi \circ T - \chi, \quad m, \chi \in L^p, \quad Pm = 0. \tag{2}$$

The last condition means that $E(m|T^{-1}\mathcal{M}) = 0$ where $\mathcal{M}$ denotes the underlying σ-algebra. It follows that $\{m \circ T^j,\ j \geq 0\}$ is a sequence of "reverse" martingale differences. Passing to the natural extension of T, it is easy to show that (1) still holds.

In the situation of (1), it is often assumed that v lies in L^∞ even though m and χ are only L^p. In such situations, $\| \sum_{j=0}^{n-1} v \circ T^j \|_q$ is well-defined for all q and it is natural to ask for which q is it the case that $\sqrt{n}$ growth holds. Using an inequality of Rio [8], it was shown in [5] that $\sqrt{n}$ growth holds for $q = 2p$ and that this is optimal. Moreover, it suffices that $p \geq 1$. To summarise,

Proposition [5]. Suppose that $v \in L^\infty$ has mean zero and that the decomposition (2) holds for some $p \geq 1$. Then there is a constant $C = C_v > 0$ such that

$$\Big\| \sum_{j=0}^{n-1} v \circ T^j \Big\|_{2p} \leq C n^{1/2}, \quad \text{for all } n \geq 1.$$

Iterated moment estimates

Now let $v_1, v_2 : X \to \mathbb{R}$ be a pair of mean zero L^∞ observables admitting a decomposition of the form (2). We are interested in the moments of the iterated sum $I_n = \sum_{0 \leq i < j < n} v_1 \circ T^i\, v_2 \circ T^j$. It is straightforward to show that $\|I_n\|_{p/2} \leq Cn$ provided $p \geq 4$, and using Rio's inequality it is shown in [3, Proposition 7.1] that $\|I_n\|_{2p/3} \leq Cn$ provided $p \geq 3$. It seems unlikely that this is optimal.

Conjecture [3]. Suppose that $v_1, v_2 \in L^\infty$ have mean zero and satisfy the decomposition (2) for some $p \geq 2$. Then there is a constant $C = C_{v_1,v_2} > 0$ such that

$$\Big\| \sum_{0 \leq i < j < n} v_1 \circ T^i\, v_2 \circ T^j \Big\|_p \leq Cn, \quad \text{for all } n \geq 1.$$

A successful resolution of this conjecture would extend the scope of the results on homogenization in [3]. In [3, Corollaries 9.2 and 9.4], it would suffice that $p > 3$, whereas currently we require $p > 9/2$. In the situation of intermittent maps ([3, Example 10.3]), the results currently valid for $\alpha < \frac{2}{11}$ would be valid for $\alpha < \frac{1}{4}$.

3. Julian Newman:

Suppose we have a standard measurable space (X, Σ), a probability space $(I, \mathcal{I}, \nu)$ and an I-indexed family $(f_\alpha)_{\alpha \in I}$ of functions $f_\alpha : X \to X$ such that the map $(\alpha, x) \mapsto f_\alpha(x)$ is $(\mathcal{I} \otimes \Sigma, \Sigma)$-measurable. Let ρ be a probability measure on X which is stationary with respect to the transition kernel

$$P(x, A) \ = \ \nu(\alpha \in I \, : \, f_\alpha(x) \in A).$$

For each $n \in \mathbb{N}$, let Q_n be the image measure of $\nu^{\otimes n}$ under the measure-valued map $(\alpha_1, \ldots, \alpha_n) \mapsto (f_{\alpha_n} \circ \ldots \circ f_{\alpha_1})_* \rho$. It can be shown that there exists a probability measure Q on the space of probability measures on X such that, given any separable metrisable topology on X generating Σ, Q_n converges to Q in the narrow topology of the narrow topology of X. One can then show that the probability measure $\rho^{(2)}$ on $X \times X$ given by

$$\rho^{(2)}(B) \; = \; \int_{Pr(X)} \mu{\otimes}\mu(B)\, Q(d\mu)$$

is stationary with respect to the two-point transition kernel

$$P^{(2)}(x, y, B) \; = \; \nu(\alpha \in I \, : \, (f_\alpha(x), f_\alpha(y)) \in B).$$

Now it is well-known that for deterministic dynamical systems, an invariant measure μ is weakly mixing if and only if $\mu \otimes \mu$ is ergodic with respect to the two-point motion. Our question is: if $\rho^{(2)}$ is ergodic with respect to the transition kernel $P^{(2)}$, does it follow that ρ is weakly mixing with respect to the transition kernel P?

4. Charlene Kalle asked a question about the invariant measure of a dynamical system arising in the study of some hybrid continued fraction algorithms. This system can be seen as a random combination of the regular continued fraction transformation (or Gauss map) and the backwards continued fraction transformation (or Rényi map). Specifically the question asks whether the absolutely continuous invariant measure of this system has a piecewise analytic invariant density.

5. Andy Hammerlindl:

   Convergence of the spectrum for skew products. Is it true that for certain skew products over expanding maps the spectrum of the transfer operator converges in some sense as the skew product converges to a direct product?

Besides the formal open problem session, several talks included open problems and conjectures.

6. In their joint talk, Gess and Scheutzow posed the following problem: does an SDE of gradient type with additive noise which admits an invariant probability measure always have a negative Lyapunov exponent (or does it at least satisfy the slightly weaker form of local asymptotic stability introduced in the talk) ? (The slides are available online at the workshop website.) Another open problem which came up during the discussion of the talk was: does the Lorenz system (which is not of gradient type) with additive noise have a positive Lyapunov exponent for small noise intensity and a negative one for large noise intensity? Simulations suggest that the answer is 'yes'.

7. Z. Brzezniak asked whether there exists conditions on the noise implying that the invariant measure for the 2-d stochastic Navier Stokes equations in unbounded domains satisfying the Poincaré inequality is unique (even in the additive noise case).

8. M. Nicol has posed the following question. Suppose $T : X \to X$ is an ergodic transformation of (X, μ).

   Borel-Cantelli Properties: Given a sequence of sets (A_n) (balls, rectangles,...) such that $A_j \in X$ and $\sum_j \mu(A_j) = \infty$, does $T^n(x) \in A_n$ infinitely often (i. o) for μ a. e. $x \in X$? If so, is there a quantitative rate?

   We let

$$S_n(x) = \sum_{j=0}^{n-1} 1_{A_j}(T^j x)$$

$$E_n = \sum_{j=0}^{n-1} \mu(A_n)$$

We say a sequence of sets (A_n) satisfies the Strong Borel-Cantelli (SBC) property if $\sum_{j=0}^{\infty} \mu(A_j) = \infty$ and

$$\lim_{n\to\infty} \frac{S_n(x)}{E_n} = 1$$

for μ a. e. $x \in X$.

A sequence of sets (A_n) satisfies the Borel-Cantelli (BC) property if $S_n(x)$ is unbounded for μ a. e. $x \in X$.

Question: Are there examples of ergodic dynamical systems and sequences of balls or rectangles which are BC but not SBC? What conditions entail BC if and only if SBC?

9. Lian and Lu posed the problem of proving a multiplicative ergodic theorem, which is valid in non-separable Banach spaces. This would be an extension of their recent monograph [4] in which they proved an MET where instead of a matrix for each point of the base system, one has an operator on a Banach space. This would likely also extend work of González-Tokman and Quas [2]. Of special interest in applications is the situation where the Banach space is L^{∞}, as it is important for PDE applications.

3 Presentation Highlights

The workshop included 20 talks overviewing recent progress and challenges around theoretical, applied and numerical sides of the subjects of random dynamical systems and multiplicative ergodic theorems.

Kening Lu (Brigham Young University) gave a talk on Entropy, Chaos and weak Horseshoes for Infinite Dimensional Random Dynamical Systems. In particular, he gave an answer to a problem on characterizing the chaotic behavior of orbits topologically or geometrically in the presence of only positive entropy for infinite dimensional dynamical systems. He showed that if a random dynamical system has a compact random invariant set such as random attractor with positive topological entropy, then the system is chaotic and has a weak horseshoe. As a corollary, he presented the same conclusion for a deterministic dynamical system with a compact invariant set of positive topological entropy. The chaotic behavior here is due to the positive entropy, not the randomness of the system. This is a joint work with Wen Huang.

Ian Melbourne (University of Warwick) talked about stochastic limits for deterministic fast-slow systems of the form

$$\dot{x} = a(x, y) + \epsilon^{-1} b(x, y), \dot{y} = \epsilon^{-2} g(y),$$

where it is assumed that b averages to zero under the fast flow generated by g. Here $x \in \mathbb{R}^d$ and y lies in a compact manifold. He presented conditions under which solutions to the slow equations converge to solutions of a d-dimensional stochastic differential equation as $\epsilon \to 0$. The limiting SDE is given explicitly. The underlying theory applies when the fast flow is Anosov or Axiom A, as well as to a large class of nonuniformly hyperbolic fast flows (including the one defined by the well-known Lorenz equations), and the main results do not require any mixing assumptions on the fast flow. This is joint work with David Kelly and combines methods from smooth ergodic theory with methods from rough path theory.

Gary Froyland (University of New South Wales) talked about existence, stability, and applications of Oseledets splittings for semi-invertible linear cocycles. He reported on a program of work to establish existence and stability results for linear cocycles in the semi-invertible situation - where the driving mechanism is invertible, but the linear actions may be non-injective - and to create numerical methods to apply to real-world models and data. The "existence of Oseledets splitting" results provide a stronger multiplicative ergodic theorem than the "classical" theorems, which only guarantee the existence of measurable Oseledets filtrations. The stability results concern continuity properties of the Lyapunov exponents and their corresponding splitting elements when the linear actions are subjected to a variety of perturbations. The applied motivations for this work are the detection and tracking of so-called coherent structures in time-dependent dynamical systems, and I will also report on the application of these constructions to fluid flow in the ocean and atmosphere. This is joint work with Cecilia González Tokman, Christian Horenkamp, Simon Lloyd, Adam Monahan, Anthony Quas, Vincent Rossi, Naratip Santitissadeekorn, Alex Sen Gupta, and Erik van Sebille.

Matthew Nicol (University of Houston) presented recent results on annealed and quenched limit theorems for random expanding dynamical systems. In particular he discussed results on annealed and quenched versions of a central limit theorem, a large deviation principle, a local limit theorem, and Erdös- Renyi type limit laws. This is joint work with Romain Aimino (Aix Marseille Universite) and Sandro Vaienti (CPT Luminy).

Chris Bose (University of Victoria) presented asymptotics for random intermittent maps in the context of expanding interval maps with a neutral fixed point. These are examples of nonuniformly hyperbolic systems which are frequently studied for their potential to give interesting statistical behaviour such as sub-exponential decay of correlation, intermittency or so-called anomalous diffusion (different terms that amount to essentially the same thing: slow relaxation to equilibrium). A random map (skew product with a Bernoulli shift) constructed from a family of such nonuniformly hyperbolic maps undoubtedly inherits some of these intermittency features, but exactly how they combine may not be immediately obvious. He showed, among other results, that the rate of correlation decay is completely determined by the 'least nonuniformly hyperbolic' map in the family, no matter how infrequently the map is chosen in the randomization. This talk reports on joint work with Wael Bahsoun and Yuejiao Duan, University of Loughborough, UK.

Andrew Török (University of Houston) presented an almost sure invariance principle for sequential and non-stationary dynamical systems. This strong form of approximation by Brownian motion was shown to hold in various examples, including observations on sequential expanding maps, perturbed dynamical systems, non-stationary sequences of functions on hyperbolic systems as well as applications to the shrinking target problem in expanding systems. (Authors: Nicolai Haydn (USC), Matthew Nicol (UH), Andrew Török (UH), Sandro Vaienti (Marseille).)

Ian Morris (Surrey) talked about the transfer operator for the binary Euclidean algorithm. This algorithm is a modification of the classical Euclidean algorithm which replaces division by an arbitrary integer with division by powers of two only. Statistical properties of the classical Euclidean algorithm – such as the average number of steps required to process a pair of integers both of which are less than N – can be studied via the thermodynamic formalism of the Gauss map acting on the unit interval. To investigate similar properties for the binary Euclidean algorithm one must instead study the thermodynamic formalism of an IID random dynamical system on the interval. He described a recent result on the transfer operator of the binary Euclidean algorithm which can be applied to resolve conjectures of R.P. Brent, B. Valleé and D.E. Knuth.

Dalia Terhesiu (University of Vienna) presented a renewal scheme for non uniformly hyperbolic flows. In recent work, I. Melbourne and D. Terhesiu, 2014 obtain optimal results for the asymptotic of the correlation function associated with both finite and infinite measure preserving suspension semiflows over Gibbs Markov maps. The involved observables are supported on a thickened Poincaré section. In more recent work with H. Bruin, a different renewal scheme for suspension flows over non uniformly hyperbolic maps is investigated by inducing to a well chosen region Y of the same dimension as the manifold (on which the flow is defined). By forcing expansion on the flow direction, they can ensure that the induced version of the flow is a hyperbolic map F. Combined with the type of renewal equation established in Melbourne and Terhesiu, 2014 and several abstract assumptions on the hyperbolic map F (and thus on the underlying map of the suspension flow), this scheme is used to estimate the correlation function of observables supported on the whole region Y.

Jairo Bochi (Pontificia Universidad Católica de Chile) talked about optimization of Lyapunov exponents of matrix cocycles. The main result presented says that if a 2x2 one-step cocycle has certain hyperbolicity properties (namely, there exist strictly invariant cones whose images do not overlap) then the Lyapunov-optimizing measures have zero entropy. The proof has two steps: first, a generalization of the Barabanov norm (similar to Mañé lemma) and second, a study of geometrical constraints between the invariant directions.

Christoph Kawan (New York University) talked about entropy for control problems and random escape rates. Namely, this talk discussed the control-theoretic problem to determine the smallest rate of information in a feedback loop above which a control system can solve a given control task. Such minimal data rates can be described by quantities that resemble topological entropy. For the control problem to render a given subset of the state space invariant, the associated entropy is related to escape rates of random dynamical systems which arise by putting shift-invariant measures on the space of admissible control functions. Here the multiplicative ergodic theorem comes into play, which allows to estimate the escape rates in terms of Lyapunov exponents.

Martin Rasmussen (Imperial College) talked about bifurcations of random dynamical systems. Despite its importance for applications, relatively little progress has been made towards the development of a bifurcation theory for random dynamical systems. In this talk, it was demonstrated that adding noise to a deterministic mapping with a pitchfork bifurcation does not destroy the bifurcation, but leads to two different types of bifurcations. The first bifurcation is characterized by a breakdown of uniform attraction, while the second bifurcation can be described topologically. None of these bifurcations correspond to a change of sign of the Lyapunov exponents, but it was explained that these bifurcations can be characterized by qualitative changes in the dichotomy spectrum and collisions of attractor-repeller pairs. This is joint work with M. Callaway, T.S. Doan, J.S.W Lamb (Imperial College) and C.S. Rodrigues (MPI Leipzig).

Anna Cherubini (University of Salento) discussed attractors for nonautonomous random dynamical systems with an application to stochastic resonance. She considered random dynamical systems with nonautonomous deterministic forcing and provide existence results for nonautonomous random attractors. In particular, she proved the existence of an attracting random periodic orbit for a class of one-dimensional random dynamical systems with a time-periodic forcing, generalising results obtained by Hans Crauel and Franco Flandoli. As an application, she discussed a standard model for the stochastic resonance, given by the one-dimensional 'overdamped' approximation of the stochastic Duffing oscillator. (Joint work with J.S.W. Lamb, M. Rasmussen and Y. Sato.)

Zdzislaw Brzezniak (York University) talked about invariant measures stochastic Navier-Stokes equations in unbounded domains via bw-Feller property. He described a general result on the existence of invariant measures for Markov processes having the so-called bw-Feller property and showed how this can be applied to stochastic Navier-Stokes equations in unbounded domains. This talk is based on joint works with M. Ondrejat and Ela Motyl. The results presented are in some sense generalisations of related results for stochastic nonlinear beam and wave equations (where a Pritchard-Zabczyk trick playes an essential rôle) obtained in a joint woth with M. Ondrejat and J. Seidler.

Benjamin Gess (University of Chicago) and Michael Scheutzow (Technische Universität Berlin) gave a joint presentation on recent results about synchronization by noise. They introduced sufficient conditions under which weak random attractors for random dynamical systems consist of single random points. These conditions focus on SDE with additive noise, for which they are also essentially necessary. In addition, they identified sufficient conditions for the existence of a minimal weak point random attractor consisting of a single random point.

As a model example, they proved synchronization by noise for an SDE with drift given by a (multidimensional) double-well potential and additive noise. While similar results are well-known in one dimension, these make essential use of monotonicity, which is is not available in higher dimensions. Another key simplifying assumption, that of convexity of potential, is also unavailable in this problem.

This work raises a number of interesting questions because the mechanism for synchronization exhibited in their work has extremely low (but still positive!) probability at small values of the noise parameter. Simulations indicate that the mechanism for synchronization that they study (while sufficient to show that synchronization will eventually take place) is different from the one that the system uses in practice. A number of very interesting questions were raised about the true synchronization mechanism, its time to occur. Tantalizingly, this suggests that in this example, the time taken for weak synchronization (any two fixed points are close w.v.h.p.) is much lower than the time taken for strong synchronization (any compact region is contracted to arbitrarily small diameter w.v.h.p.). This is joint work with Franco Flandoli.

Peter Imkeller (HU Berlin) talked about the dynamics of the Chafee-Infante equation with Lévy noise. Dynamical systems of the reaction-diffusion type with small noise have been instrumental to explain basic features of the dynamics of paleo-climate data. For instance, a spectral analysis of Greenland ice time series performed at the end of the 1990s representing average temperatures during the last ice age suggest an $\alpha-$stable noise component with an $\alpha \sim$ 1:75: The model of the time series consisted of a dynamical system perturbed by α-stable noise, and he introduced an efficient testing method for the best fitting α. A class of reaction-diffusion equations with additive α-stable Lévy noise (a stochastic perturbation of the Chafee-Infante equation) was introduced, in order as a generalization of the solution of this model selection problem. He described a study of exit and transition between meta-stable states of their solutions. (Joint work with A. Debussche, J. Gairing, C. Hein, M. Högele, I. Pavlyukevich)

Szymon Peszat (Institute of Mathematics, Jagiellonian University and Institute of Mathematics, Polish Academy of Sciences) spoke about time regularity of solutions to SPDEs. When driven by a Wiener process, this regularity can be studied using either Kolmogorov criterion, Kotelenez theorem or Da Prato-Kwapień-Zabczyk factorization. It turns out that very often the solution is continuous is a given state space E even if the noise takes values in a bigger space $U \hookleftarrow E$. If the noise is of jump type and does not take values in the space space then typically the solution is not càdlàg. In fact during the talk different concepts of càdlàg property were discussed. A special emphasis was put on infinite systems of linear equations driven by independent Lévy processes. The talk was based on the papers [6, 7, 1].

Barbara Gentz (University of Bielefeld) talked about the effect of noise on mixed-mode oscillations. Many neuronal systems and models display so-called mixed-mode oscillations (MMOs) consisting of small-amplitude oscillations alternating with large-amplitude oscillations. Different mechanisms have been identified which may cause this type of behaviour. In her talk, she focused on MMOs in a slow-fast dynamical system with one fast and two slow variables, containing a folded-node singularity. The main question she addressed whas whether and how noise may change the dynamics.

After outlining a general approach to stochastic slow-fast systems, she discussed how to apply this method to the model system, showing the existence of a critical noise intensity beyond which the small-amplitude oscillations become hidden by noise. Furthermore, she showed that in the presence of noise sample paths are likely to jump away from so-called canard solutions earlier than the corresponding deterministic orbits. This early-jump mechanism can drastically change the mixed-mode patterns, even for rather small noise intensities. The methods used to derive the results range from deterministic bifurcation theory and averaging to martingale techniques and estimates on Markov transition kernels. Joint work with Nils Berglund (Université dOrléans) and Christian Kuehn (TU Wien).

Francesco Ginelli (University of Aberdeen) talked about characterizing dynamics with covariant Lyapunov vectors (CLVs), or Oseledets splittings, which have an important tool for characterizing chaotic dynamics in high dimensional systems. CLVs define an intrinsic, non orthogonal basis at each point in phase space which is equivariant with the dynamics. He presented details of the dynamical algorithm he introduced to efficiently compute CLV's. He also discussed its numerical performance and compared it with other algorithms presented in the literature. He presented selected applications of CLV's to characterize the collective dynamics of globally coupled systems, to quantify the degree of hyperbolicity, and to evaluate the number of effective degrees of freedom in chaotic, spatially extended dissipative systems such as the Kuramoto-Sivashinsky equation.

There were also two student talks: Julian Newman (Imperial College) presented necessary and sufficient conditions for stable synchronisation in random dynamical systems, and Joseph Horan (University of Victoria) talked about triangularizability in the Multiplicative Ergodic Theorem.

4 Scientific Progress Made and Comments from Participants

- Dalia Terhesiu has discussed research visits of two workshop participants to Vienna. In March, 13 or April 15 2015 Ian Morris will give a talk in the Budapest-Wien dynamical seminar. Gary Froyland has been invited to give a talk in the weekly dynamical seminar organized at Vienna University in June/July 2015.
- Terhesiu and Melboune established some research steps on the topic of mixing for the periodic geodesic flow.
- Brzezniak started a collaboration with Chojnowska-Michalik about the generalisations of the results presented by Imkeller to the 2-d domains (for instance a sphere S^2).
- Brzezniak discussed with Peszat a possibility of restarting their work (joint with R Tribe) about the 2-d Anderson model with space time white noise.
- Scheutzow and Brzezniak discussed briefly their ongoing project on stabilization by noise for linear SPDEs in Hilbert spaces.
- Brzezniak and González-Tokman discussed about prospects of extending results on multiplicative ergodic theorems and Oseledets splittings for non-invertible maps to the case of SPDEs.
- Horan mentioned: "talking to Ian Morris helped guide me to some previously unknown literature on the subject of my research. This was extremely invaluable, for both motivating and placing my research in the field. As well, the wide variety of talks helped introduce me to other areas of dynamical systems, which helps me to get a better picture of what's out there and what I might wish to research".

- Charlene Kalle is using the techniques presented by Matt Nicol, to see if she can prove CLT, large deviations and dynamical Borel-Cantelli lemmas for a random composition of Gauss like maps. Nicol mentions: "It will need some modification of our proofs as the partition on which the maps are piecewise C^2 is infinite. However recent work of Inoue proofs a spectral gap so some of the techniques I spoke about (my joint work with Aimino and Vaienti) will almost certainly work".
- Scheutzow states: "I had fruitful scientific discussions with Martin Rasmussen, Maximilian Engel and Julian Newman about questions related to synchronization by noise and Hopf bifurcations. I agreed with Martin on a continued exchange of ideas in the future (I invited him to come to Berlin for a few days and he invited me to come to Imperial College for a few days later this year). I had discussions with Kening Lu on possible generalizations of the concept of entropy. I had an interesting discussion with Anthony Quas about the order of magnitude of the time until synchronization takes place when the noise intensity of an sde tends to 0. I discussed with Hans Crauel and Benjamin Gess about invariant measures for RDS which are not of white noise type (it is possible that this will initiate a collaboration resulting in a joint paper)."
- Bose writes: "As someone who works mostly on the theory side, I found the excellent presentations on applications to be really inspiring. Specific high points for me:
  - Matt Nicol's talk. I was really intrigued by his problem on chilled limit theorems and the technical obstructions that he identified. We talked briefly about this. It was something I was not aware of prior to this conference. Matt made it very clear what the problem is, and what needs to be done.
  - Ian Melbourne's presentation on fast-slow systems. As always, a masterful and inspiring presentation. Although I've dabbled in random maps for a number of years, I was not aware of the nice applications that are going on out there. Ian's talk bridged the gap between theoretical results and implications for application.
  - Dalia Terhesiu's talk, and a short discussion we had afterward. The kinds of problems she reported on are very close to things I have been thinking about. Dalia will be in Loughborough later this spring, at a workshop organized by my co-author Bahsoun and we were kind of hoping to pick up the conversation again, through Bahsoun if I cannot be there in person."
- Julian Newman and Anthony Quas solved Thomas Kaijser's open problem 1, to decide whether there exists a bi-Lipschitz iterated function system on a compact metric space with uniformly transitive dynamics and yet with more than one stationary probability measure. (They found, independently, that such systems do exist.)
- Newman also presented a problem during his talk, which Anthony Quas then solved. The question was to prove or disprove a formula for the size of the random attractor of the RDS generated by a circle map of degree one perturbed by conjugation with a random rotation selected with uniform distribution.
- Gottwald stated: "I thoroughly enjoyed it and learned a lot from the talks and the many discussions I had".
- Blumenthal, González-Tokman and Quas discussed various proof strategies and techniques for establishing multiplicative ergodic theorems on Banach spaces. Such discussions are likely to lead to future collaborations connected to Lian and Lu's problem 9.
- Török invited González-Tokman to give a seminar at the University of Houston in February.
- Nicol and González-Tokman discussed the conditions for the quenched CLT presented in Nicol's talk. The discussion is likely to continue during González Tokman's visit to the University of Houston.
- Froyland and González-Tokman continued their ongoing discussion on finding more efficient ways of using Oseledets splitting information to identify coherent structures from models and data coming from geophysical flows.
- Doan and González-Tokman talked about various open problems regarding Lyapunov exponents for operator cocycles on Banach spaces. This discussion will likely develop into collaborative work.

- Kalle and González-Tokman initiated a discussion on metastability questions, and about the possibility of using more probabilistic tools in the deterministic setting.
- Before the workshop started, some people indicated to us that they would be interested in watching the talks online. Alexandra Neamtu (Jena) commented afterwards: "It was a great workshop and I am very happy that I could watch the talks and learn a lot."

5 Outcome of the Meeting

BIRS provided a wonderful atmosphere for research and scientific collaboration. New research connections have been established among the communities of random, non-autonomous and deterministic dynamical systems. Several scientific collaborations have been initiated and progressed during the workshop week.

The scientific talks presented at the workshop will remain available for participants and other interested scientists at the workshop website **http://www.birs.ca/events/2015/5-day-workshops/15w5059**. During the workshop, talks were recorded using BIRS video facilities. Furthermore, slides of most talks – all except those presented at the board or with the document camera – are available at the workshop website.

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Multiplicative ergodic theorems: In the dynamical systems literature, for many years the focus was restricted to dynamical systems where the noise was independent and identically distributed. In practice, this means that one has a family of maps, and at each stage a map is randomly selected and applied to give a new point of the orbit.Â This is no longer true if the maps are not applied in an i.i.d. manner. Since the assumption of i.i.d. random maps is not reasonable in many physical situations (e.g. where a dynamical system is forced by a slowly moving much more massive dynamical system), there was a substantial gap between the theory and desired applications. 1 Non-Autonomous Dynamical Systems 1.1 Introduction . . . 1.2 Non autonomous dynamical systems: deâ€¦nition and examples . . . 1.3 Invariance. Criticalities of the concept for NADS . . .Â 1.2. non autonomous dynamical systems: definition and EXAMPLES3. is a particular case, called simply a Dynamical System (DS). The correspondence with the more general concept is.Â The compact global attractor constructed by Theorem 20 is invariant in this new sense, when the NADS comes from a cocycle. Theorem 40 Let 't (!) be a cocycle over ; (t)t2R , and let. U! (s; t) = 't s (s!) be the associated NADS. The Oseledets multiplicative ergodic theorem is a basic result with numerous applications throughout dynamical systems. These notes provide an introduction to this theorem, as well as subsequent generalizations. They are based on lectures at summer schools in Brazil, France, and Russia. Type. Survey Article. Information. Ergodic Theory and Dynamical Systems , Volume 39 , Issue 5 , May 2019 , pp. 1153 - 1189. DOI: https://doi.org/10.1017/etds.2017.68[Opens in a new window]. Copyright. Discontinuous random dynamical systems or cocycles (Definition 2.1) generated by stochastic differential equations (SDEs) with LÃ©vy processes have attracted attention more recently [14,13,18,19]. In this paper, we consider topological equivalence between discontinuous cocycles, generated either by SDEs with LÃ©vy processes or by related differential equations with random coefficients (i.e. Topological equivalence for discontinuous random dynamical systems and applications. Article. Full-text available.Â The transformation from stochastic differential equations to random integral equations and a multiplicative ergodic theorem are used. Moreover, the result is applied to physical and economic problems. View full-text.