On the origins of the idea of the multiplicative decomposition of the deformation gradient*

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The objective of this short note is to trace back the major contributions that led to the multiplicative decomposition of the deformation gradient in finite plasticity, nonlinear thermoelasticity, and growth mechanics. In the 1940s, Eckart in the US and Kondo in Japan, independently paved the road to the formulation of a nonlinear theory capable of modeling anelastic phenomena. As opposed to assuming, for a given body, the existence of a global stress-free configuration (the "principle of relaxability-in-the-large" according to Eckart) that the body takes whenever it is completely relaxed, Eckart [1948] suggested an alternative framework for anelasticity based on what he called "relaxability-in-the-small". He conceptually constructed a local stress-free "fragmented" state following a local relaxation of the reference configuration by "cutting out" a "small bit of matter" around every material point and letting it relax independently of the remainder of the body. He also asserted that such a construction should be accompanied by an elastic deformation to ensure that the body keeps its structural integrity. This is nothing but the decomposition of the deformation gradient into an anelastic relaxation, leading to the so-called "intermediate" configuration, followed by the elastic portion of the deformation gradient.

Independently of Eckart's work, Kondo [1949] observed that due to plastic deformations, the relaxed state of a body has a non-trivial geometry that is not compatible with that of the Euclidean space. This observation first led him to construct a stress-free configuration as a Riemannian manifold in which a non-vanishing curvature is a measure of the incompatibility of the plastic deformation. Inspired by the works of Cartan [1926, 1928] on non-trivial holonomy groups, Kondo [1950a,b, 1952] extended his framework to consider the material body as a non-Riemannian space with a non-zero torsion. He used this geometric framework in the context of crystals with geometrical imperfections, e.g. dislocations, and introduced the idea of considering the stress-free state as "an amorphous aggregation" of small pieces of relaxed perfect "crystalline pieces" that he modeled as a non-Riemannian manifold. Further, he interpreted the torsion tensor as a measure of the density of dislocations and initiated the development of a geometric theory of dislocation mechanics. Soon after, further contributions to the nonlinear theory of dislocation mechanics were introduced by Kröner [1955], Kröner and Seeger [1959], and Bilby et al. [1955]. For a review of the interactions between the Japanese (led by Kondo), the British (led by Bilby), and the German (led by Kröner) schools and their contributions, see Kondo [1964]. It is worth mentioning that Sedov [1965] independently realized that a body in

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plastic deformation can be relaxed in a stress-free intermediate configuration, which he called “a new starting position”, with a changing metric that is generally non-Euclidean.

Following the original idea of local relaxation inspired by the pioneering works cited above, the first formal introduction of the multiplicative decomposition of the deformation gradient in finite plasticity appeared in the late 1950s in the work of Bilby et al. [1957]. Bilby et al. [1957] called the total deformation gradient $\mathbf{F}$, the elastic deformation gradient $\mathbf{F}_e$, and the plastic deformation gradient $\mathbf{F}_p$, “shape deformation, “lattice deformation”, and “dislocation deformation”, respectively. The decomposition $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ was explicitly written in [Bilby et al., 1957, Page 41, Eq. (12)]. The same decomposition is seen in [Kröner, 1959, Page 286, Eq. (4)] as well. Almost a decade later, Lee and Liu [1967] and Lee [1969] discussed the multiplicative decomposition in finite plasticity and received most of the credit for it. In nonlinear thermoelasticity, the first formal introduction of the multiplicative decomposition of the deformation gradient is due to Stojanović et al. [1964] and Stojanović [1969]. In the biomechanics and growth mechanics literature, the introduction of the multiplicative decomposition is usually attributed to Rodriguez et al. [1994]. However, it was first introduced about a decade earlier independently in Russia by Kondaurov and Nikitin [1987] and in Japan by Takamizawa and Hayashi [1987], Takamizawa and Matsuda [1990], and Takamizawa [1991].

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One important use of the deformation gradient is that it specifies how the volume is changed during a transformation. If dV is a small volume element in the reference configuration, then that volume element is transformed into the volume element. (4.79).Â The proof of this theorem is given in different text books [2, 10] and is here left as an exercise. The deformation of a body can be divided into different classes depending on the structure of the deformation gradient: â€¢ If F does not vary from location to location in a body then the deformation is said to be homogeneous. â€¢ If F(X) is a function of the position then the deformation is said to be inhomogeneous. â€¢ If. J=detF=1. then the deformation is said to be isochoric. 4.5.1 Eigenvalue and Spectral Decompositions. An explanation on the origin of residual stresses in living bio-tissues was rst presented theoretically by Rodriguez et al. (1994) via the multiplicative decomposition (MD) method. They showed that residual stresses in a bio-tissue are created by heterogeneous growth and can be calculated from a given growth gradient tensor. The constrained growth deformation is decomposed into unconstrained growth deformation and pure elastic deformation (Figure 1) with the relation F = FeFg, where F is the total deformation, Fe the pure elastic deformation, and Fg the growth deformation. From a modeling stan... Rate-Independent Deformation Model Based on Multiplicative Decomposition of the Deformation Gradient. Principal of virtual power. Consequences of frame indifference.Â homogeneous) plastic deformation up to 20% (or more) on the incoming stack of polymeric lms. The roller-axes should be oriented vertically (since the incoming lms emerge from an upstream step with their initial thickness-wise dimension oriented in the horizontal direction, Figure 1). (ii) The desired levels of plastic strains should be achieved in the incoming stack of folded lms; being fed at linear rate of 5 âˆ' 30 mm/min.