Conductance: from electrical networks, through graphs, to dynamical systems

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Abstract—*We introduce the notion of conductance in dynamical systems on iterated maps of the interval. Our starting point is the notion of conductance in the graph theory. We pretend to apply the known results in this new context.*

I. Introduction

The transfer of concepts from one area of knowledge to another has been serving as impulsive force of the development of mathematics. It was so with the entropy, which arrived from the thermodynamic and was brought by Kolmogorov to the dynamical systems. Today, thanks to Sinai, Adler, Konheim, McAndrew, Misiurewwitz, Szlenk and others, this concept is commonly used and calculated in this area, see [6].

Our objective with this paper is to introduce the notion of conductance in the discrete dynamical system on the iterated map of the interval. We will use the Markov partition of the interval to define an associated graph. By analogy with the electric circuits, the conductance was defined and is known for a regular graph without orientation. We will extend this definition to a more general setting to include both non-regular and oriented graphs, which are more useful to represent our dynamical systems. Then, we go back, and bring the definition of conductance with us to use in the study of certain families of maps, which cannot be differentiated by the topological entropy.

II. Conductance of a discrete dynamical system

In this section we will introduce formally, the notion of conductance of a dynamical system. First we will introduce this concept in the graph theory, where it has been studied for several years, see [1] for more details.

A. Graphs

An *unordered graph* G is an ordered pair of sets (V,E) , such that E is a subset of $V \times V$ of unordered pairs of V. We will call V the set of *vertices* and E the set of *edges*. An edge $\{i,j\}$ is said to join the vertices i and j and is denoted by ij. The *order* of G is the number of vertices in G and is denoted by $|G|$. The *degree* of a vertex i is the number of edges in E, which one of the "endpoints" is i, that is, the number of elements of E, for which i is one of the two components. A graph is said $k-$*regular* if each vertex has degree k, for some k and is said connected if there isn't any isolated vertex (every vertex has an edge with him as endpoint). To each unordered, connected k-regular graph G with n vertices $\{1,2,...,n\}$, is associated a *simple random walk* $X=(x_t)_{t=0}^{\infty}$ in its simplest form: starting at x_0, its next vertex, x_1, is chosen randomly from the neighbours of x_0. Next x_2 is chosen among neighbours of x_1, and so on. Set $p_i(t) = Prob(x_t = i)$. Thus X is the simple random walk (Markov Chain) with initial distribution $p_0 = (p_1^{(0)}, p_2^{(0)}, ..., p_n^{(0)})$ and $p_t = (p_1^{(t)}, p_2^{(t)}, ..., p_n^{(t)}) = p_0 P^t$ is the distribution of x_t. We view the distributions as row vectors in $\mathbb{R}^{\mathbb{V}}$ and we call P the transition matrix.

Definition 1: A row vector $\pi \in \mathbb{R}^{\mathbb{V}}$ is a stationary distribution of the chain X with transition matrix P if

a) $\pi(i) \geq 0$, for all $i \in V$;
b) $\sum_{i\in V} \pi(i) = 1$;
c) $\pi = \pi P$

Definition 2: A Markov chain X with transition matrix P is said to be ergodic if it has a stationary distribution. Is said to be irreducible if for all $i,j \in V$, there is an m such that $(P_{ij})^m > 0$. Is said aperiodic iff for all $i \in V$, $\gcd\{m : (P_{ij})^m > 0\} = 1$.

It is known that any finite, irreducible Markov chain is ergodic.

Back to our simple random walk, we have then p_t tends to the stationary distribution, that is, in this case, the vector $\pi = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$. The measure of the speed of convergence $p_t \to \pi$, is given by the *mixing rate* of the random walks on G

$$\mu = \sup_{p_0} \lim_{t\to\infty} \sup \|p_t - \pi\|_2^{\frac{1}{t}} ,$$

where the supremum is taken over all initial distributions p_0. As is pointed out in [1], the mixing rate μ is easily described by the eigenvalues of P. By definition, $P = A/k$, where $A = (a_{ij})_{ij=1}^{n}$ is the adjacency matrix of G, defined for $i \neq j$ by the number of edges from i to j (note that an edge from i to j is also an edge from j to i) and for $i = j$, by the number of loops at i. As usual, A is identified with a linear endomorphism of the vector space $C_0(G)$ of all functions from V into $\mathbb{C}$. The matrix P is hermitian and then it has only real eigenvalues. It is known that $1 = \lambda_1 > \lambda_2 \geq ...\lambda_n > -1$ and we have that the mixing rate μ is precisely $\lambda = \max\{\lambda_2, |\lambda_n|\}$.

In fact, we shall estimate the speed of convergence to the stationary distribution in terms of the *conductance* Φ_G of a graph. The definition follows.

Definition 3: Let G be an *unordered, connected, k-regular graph*. Define the conductance Φ_G of G by

$$\Phi_G = \min_{U \subset V} \frac{e(U, \bar{U})}{d \, \min\{|U|, |\bar{U}|\}}$$

where $= V \backslash U$ and $e(U, \bar{U})$ is the number of edges from U to $\bar{U}$. Note that if $|U| \leq n/2$, as we may assume, then $k|U| = \sum_{u \in U} d(u)$ is the maximal number of edges that may leave U, so $\frac{e(U,\bar{U})}{d\,|U|}$ is the proportion of edges "leaving" U.

So, for a dynamical system described by an hermitian matrix, we can speak about the conductance introduced in the above definition.

B. Markov Chains

In a general way, to each discrete dynamical system (I, f) defined by the iterates of a map f on the interval I, we associate a Markov matrix, which is representable by a non-regular, oriented graph G_f (the elements of E are now ordered pairs). So we have systems defined by the adjacency matrix $A_f = (a_{ij})$ of G_f, that is, the $0-1$ matrix where $a_{ij} = 1$ iff ij is an edge. In turn, to each A_f, we associate a probability matrix P_f and an invariant measure (the measure of maximal entropy) π. So we have what is called a random walk in a weighted directed graph, with loops allowed, described by a transition matrix P_f, which is no longer hermitian, but can however represent an ergodic system. We can now establish the notion of conductance of a discrete dynamical system.

Definition 4: Let f be a map on the interval, $P_f = (P_{ij})_{ij=1}^{n}$ be the probability matrix associated to f and $\pi_f = (\pi_i)_{i=1}^{n}$ be the invariant measure. Define conductance of (I, f) by

$$\Phi_f = \min_{\substack{0<\pi(S)\leq 1/2 \\ S \subset V}} \frac{\sum_{i \in S, j \in \bar{S}} \pi_i \, P_{ij}}{\sum_{i \in S} \pi_i}$$

Another possible approach to the conductance is trough the discrete laplacian of a graph defined next.

Definition 5: Let $A_f = (a_{ij})_{i,j=1}^{n}$ be the adjacency matrix associated to (I, f) and G_f the Markov graph. Define the diagonal matrix $D_f = (d_{ij})_{i,j=1}^{n}$, putting in the diagonal d_{ii} the number of edges that incide (in and out) in the vertex i (loops contribute with 2). We will call the matrix

$$\Delta_f = D_f - (A_f + A_f^T)$$

the laplacian matrix of the graph G_f.

As we will see, the smallest non-zero eigenvalue of the laplacian is closely related with the conductance of the system.

III. Bimodal family

Lets consider a bimodal family of maps $S_{a,b}$ (cubic like maps, see Figure 1), depending on two parameters a and b. We want one parameter to be related with the topological entropy (defined as the logarithm of the growth number of periodic points) and use the second to distinguish the systems via conductance.

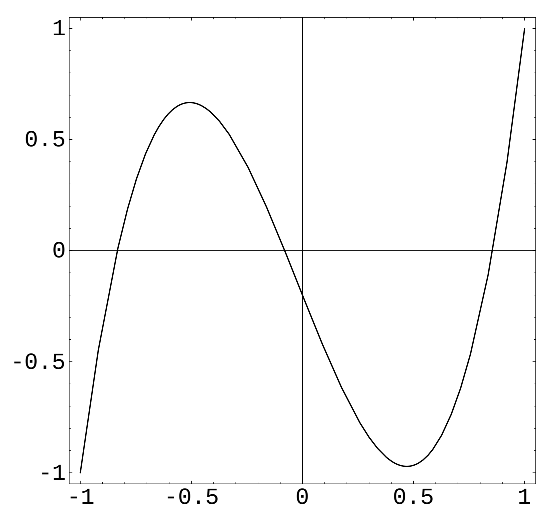

Fig. 1. Cubic like map.

For different values of the parameters a and b we obtain trajectories, which can be symbolically expressed by the itineraries of the two critical points c_1 and c_2.

If we take a point x in I, we will call the address of x, $A(x)$, one of the symbols L, A, M, B, R according to the following rule:

$$A(x) = \begin{cases} L & \text{if } x < c_1 \\ A & \text{if } x = c_1 \\ M & \text{if } x > c_1 \text{ and } x < c_2 \\ B & \text{if } x = c_2 \\ R & \text{if } x > c_2 \end{cases}$$

The itinerary of $x \in I$ will be then the sequence of symbols $A(x), A(f(x)), A(f(f(x)))\ldots.$

Using the itinerary of the critical points c_1 and c_2 we obtain a Markov partition of the interval I. The exists an uncountable quantity of different dynamic types. To introduce the study of conductance and discrete laplacian in the maps of the interval we considered 3 families, given by the pair of itineraries of the critical points:

1. $((R^kA)^\infty, (L^kB)^\infty)$, $k = 1, 2, ...$; with topological entropy $h_t(f) \in [0, \log(3)]$.
2. $((RM^kA)^\infty, (LM^kB)^\infty)$, $k = 1, 2, ...$; with topological entropy $h_t(f) \in [0, \log(2)]$.
3. $RM^k(BLM^k)^\infty$ and $RL^k(BLL^k)^\infty$, $k = 1, 2, \ldots$. Here the trajectory of c_1 falls in the trajectory of c_2; with constant topological entropy $h_t(f) = \log(2)$.

After a numerical study, we can state the following result.

Theorem 1: Let $f : I \to I$ be a piecewise monotone map. The conductance Φ_f and the first non-zero eigenvalue $\lambda_1(\Delta_f)$ of the discrete laplacian Δ_f are functions that decrease when the periods of the critical points increase, converging to a constant depending on the topological entropy of f.

This result can be proved by symbolic dynamic methods and it is illustrated in Figures 2, 3, 4 and 5,

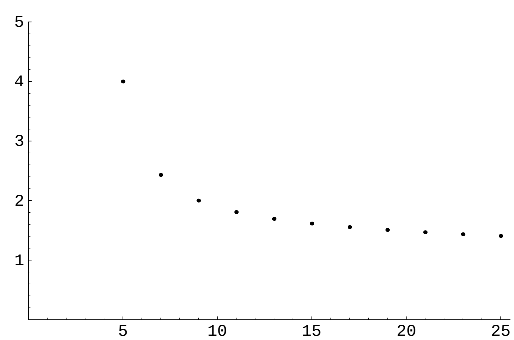

Fig. 2. The first eigenvalue $\lambda_1(\Delta)$ $(\dim A_f)$ for the family $((R^kA)^\infty, (L^kB)^\infty)$, with topological entropy $\in [0, \log(3)]$.

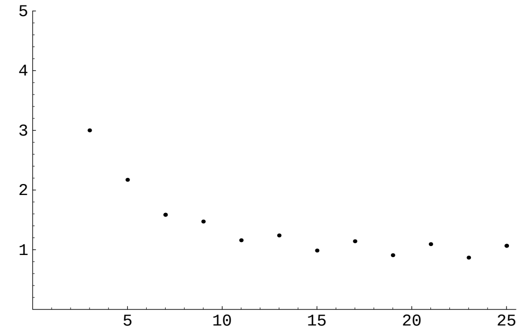

Fig. 3. The first eigenvalue $\lambda_1(\Delta)$ $(\dim A_f)$ for the family $((RM^kA)^\infty, (LM^kB)^\infty)$, with topological entropy $\in [0, \log(2)]$.

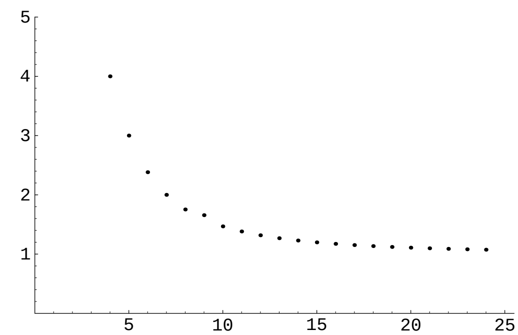

Fig. 4. The first eigenvalue $\lambda_1(\Delta)$ $(\dim A_f)$ for the family $RL^k(BLL^k)^\infty$, with constant topological entropy $= \log(2)$.

In the Figure 3 the oscillation in the decreasing of the $\lambda_1(\Delta_f)$ is due to parity of number of M symbols (because the function $f_{|M}$ has negative slope).

In the Figure 6 and Figure 7 we can see the variation of the modulus of the second eigenvalue $\lambda_2(A_f)$ of the adjacency matrix A_f increasing with the dimension of A_f, see [2] and [3] for the relation with the mixing rate.

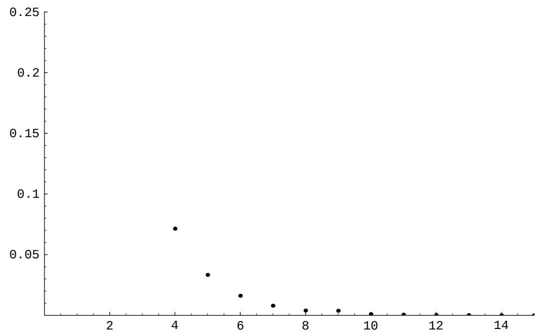

Fig. 5. The conductance Φ_f $(\dim A_f)$ for the family $RL^k(BLL^k)^\infty$, with constant topological entropy $= \log(2)$.

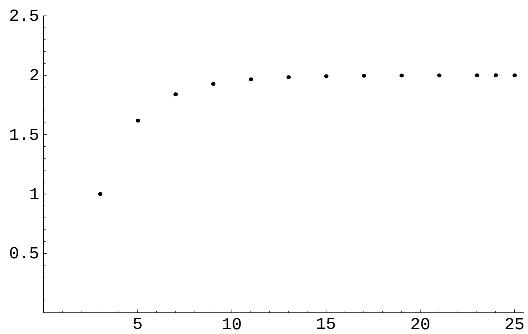

Fig. 6. The second eigenvalue $\lambda_2(A_f)$ $(\dim A_f)$ for the family $((RM^kA)^\infty, (LM^kB)^\infty)$, with topological entropy $\in [0, \log(2)]$.

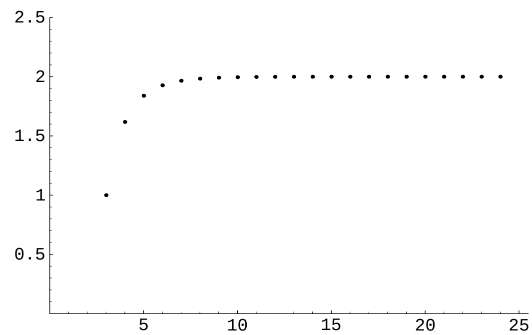

Fig. 7. The second eigenvalue $\lambda_2(A_f)$ $(\dim A_f)$ for the family $RM^k(BLM^k)^\infty$, with constant topological entropy $= \log(2)$.

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an electrical circuit to the graph induced by A. Consider a connected electrical net-work with n nodes, branch conductances Aij â‰¥ 0, and shunt conductances Aii â‰¥ 0 connecting node i to the ground. By Kirchhoâ€™s and Ohmâ€™s laws the current-balance equations I = QV are obtained, where I âˆˆ RnÃ—1 are the currents injected at the nodes, V âˆˆ RnÃ—1 are the nodal voltages, and the conductance matrix Q âˆˆ RnÃ—n. âˆ—This work was supported in part by NSF grants IIS-0904501 and CNS-0834446. â€ Florian DoÂ¨rer and Francesco Bullo are with the Center for Control, Dynamical Systems and Computation, University of Cali... Total parallel conductance is greater than any of the individual branch conductances because parallel resistors conduct better together than they would separately: To be more precise, the total conductance in a parallel circuit is equal to the sum of the individual conductances: If we know that conductance is nothing more than the mathematical reciprocal (1/x) of resistance, we can translate each term of the above formula into resistance by substituting the reciprocal of each respective conductance: Solving the above equation for total resistance (instead of the reciprocal of total resistance), we can invert (reciprocate) both sides of the equation: So, we arrive at our cryptic resistance formula at last! 6 Dynamical Systems on Dynamical Networks. 7 Other Resources. 8 Conclusion, Outlook, and Open Problems.Â Dynamical Systems on Networks. A Tutorial. Frontiers in Applied Dynamical Systems: Reviews and Tutorials.Â We focus on â€œsimpleâ€ situations that are analytically tractable, though studying more complicated systemsâ€"typically through direct numerical simulationsâ€"is also worthwhile. Many of the dynamical processes that we consider can of course be studied in much more complicated situations (including on directed networks, weighted networks, temporal networks [141], and multilayer [28, 173] networks), and many interesting new phenomena occur in these situations. system that is based on a homogenization procedure. fractal: a geometrical object that is invariant at any scale of magnication or reduction. multifractal: a generalization of a fractal in which dierent subsets of an object have dierent scaling behaviors. percolation: connectivity of a random porous network. percolation threshold pc: the transition between a connected and disconnected network as the density of links is.Â Third, the nodes voltages of a network through which a steady electrical current ows are har-monic [15]; that is, the voltage at a given node is a suitably-weighted average of the voltages at neighboring nodes.Â We will focus on how the conductance of such a network vanishes as the percolation threshold is approached from above. Electrical Conductance and Conductivity or Specific conductance of a substance. Definition and formulas for electrical conductance and conductivity.Â The more the conductance of a material is the more current passes through the material when same Potential or voltage is applied to itâ€™s terminals. Formula of Electrical Conductance: By formula Conductance is the opposite of resistance and is denoted by â€œGâ€. or