WS 2021/22
Prof. Stefan Schwede
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Graduate Seminar on Topology (S4D2)

Lie Groups and Their Representations

Tuesdays, 14ct–16

Lie groups were introduced by Sophus Lie in the 1870's in order to capture symmetries of differential equations. Nowadays, they are studied as interesting objects in their own right, which involves methods from topology, geometry, (functional) analysis, algebra, and representation theory. In this seminar we want to focus on some of the topological, group theoretic, and representation theoretic aspects of Lie groups, and in particular on classification results for several classes of Lie groups and their representations. Our main reference is [Bt85], but for several advanced topics we rely on other accounts.

The seminar takes place Tuesdays from 14:15 to 15:45. At the present point, we expect that we can have in-person seminar meetings next semester, and the place will be announced in due time.

Note. Especially for some of the later topics, the references can contain more material than you will be able to present in your talk, and you should decide on which parts you want to focus (apart from the results explicitly mentioned in the talk description below, of course). In the case of multi-part talks it is your own responsibility to get in touch with the other speaker(s) and discuss which topics they are going to present or, conversely, which notions or results they will need for their talk.

In any case, you should meet with Tobias Lenz (`lenz@math.uni-bonn.de`) *at least two weeks before your talk* to go through the material you want to present and to discuss any questions you might have.

Prerequisites

You should be familiar with basic topology, in particular with the topics of the lecture "Einführung in die Geometrie und Topologie." For the later talks, knowledge of the classical representation theory of finite groups over fields of characteristic zero will be assumed, roughly to the amount of [Lan02, XVII.1–4 and XVIII.I–5] or [tom09, 1–2].

Talks

1. **Reminder on manifolds (T Ezubova)** Oct 12, 2021
   This introductory talk should recall some basic notions concerning manifolds. In particular, you should remind us all of the definitions of topological, smooth, and analytic manifolds, smooth maps, and closed submanifolds. Afterwards, discuss several equivalent definitions of the tangent space (in particular in terms of germs of paths, of derivations of maps to $\mathbb{R}$, and as derivations of germs of such maps) and construct the tangent bundle; for the latter you should also give a brief recap of fiber bundles and vector bundles. Finally, discuss vector fields. Some of this material is spread out between [Bt85, I.1–4] (and you should follow the conventions there!), but you probably want to consider other sources as well.

2. **Lie groups and Lie algebras (G Lafazanidis)** Oct 26, 2021
   Define Lie groups and discuss several examples [Bt85, I.1]. Introduce abstract Lie algebras (over any field), and give three equivalent definitions of the Lie algebra $\mathfrak{g}$ associated to a Lie group G (in terms of vector fields, based on the tangent space at the unit, or as one-parameter groups). Finally, compute the associated Lie algebra for several examples of matrix groups [Bt85, I.2].

3. **Some first classification results (R Feller)** Nov 2, 2021
Introduce the exponential map, establish its basic properties [Bt85, I.3.1–5], and do Exercises 1 and 2 of [Bt85, I.3.13]. Afterwards, classify connected abelian and compact abelian Lie groups [Bt85, I.3.6–8]. Finally, prove Kroneckers Theorem on generators of tori and use this to classify topologically cyclic compact Lie groups [Bt85, I.4.13–14].

4. **New Lie groups from old (R Floris)** Nov 9, 2021
Introduce Lie subgroups and prove that a subgroup of a Lie group is a Lie subgroup if and only if it is closed. Conclude that all continuous homomorphisms of Lie groups are smooth [Bt85, I.3.9-12]. Afterwards, discuss quotients of Lie groups [Bt85, I.4.1–12]. If time permits, introduce coverings of Lie groups [Tit83, II] and construct the Spin groups [Bt85, I.6] (either by the general theory or as a concrete construction).

5. **The Haar integral (A Schröter)** Nov 16, 2021
The main goal of this talk is to construct a translation invariant integral on any given compact Lie group G and to establish its basic properties following [Bt85, I.5]. If time permits, mention or sketch some aspects of the theory for general locally compact topological groups, in particular existence and uniqueness; you can follow for example [vQ01, 19] or [Hal74, XI] for this part.

6. **Repr. theory of compact Lie groups I: Basic results (T Manopulo)** Nov 23, 2021
Define representations of a given compact Lie group G, introduce the analogue of the group algebra, prove the existence of a G-invariant inner product, and state a version of Maschke's Theorem in this context [Bt85, II.1]. Moreover, prove Schur's Lemma and develop the theory of characters [Bt85, II.4]. Finally, give the abstract definition of representative functions and show that they are precisely the ones arising from representations [Bt85, III.1].

7. **Repr. theory of compact Lie groups II: Examples (D Abou Ali)** Nov 30, 2021
Classify irreducible representations of compact abelian Lie groups [Bt85, II.8] and of a non-empty subset of $\{\mathrm{SU}(2), \mathrm{SO}(3), \mathrm{U}(2), \mathrm{O}(3)\}$ [Bt85, II.5]. If time permits, introduce representations of Lie algebras, discuss how a representation of a Lie group G yields a representation of the associated Lie algebra $\mathfrak{g}$, and mention the relationship between the two for simply connected G (which will be proven in a later talk); if you have even more time, discuss representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ [Bt85, II.9–10].

8. **The Peter-Weyl Theorem (G Martínez de Cestafe Pumares)** Dec 7, 2021
It might be useful to have some experience with functional analysis.
The goal of this talk is to prove the Peter-Weyl Theorem saying that the representative functions are dense in the continuous functions, and to deduce the decomposition of the algebra of matrix coefficients analogous to the decomposition of the group algebra [Bt85, III.2–3]. As applications, prove that every compact Lie group admits a finite-dimensional faithful representation and that every quotient space G/H embeds into some G-representation [Bt85, III.4].

Note. The proof of the Peter-Weyl theorem requires some functional analysis, and you should introduce the requisite notions and theorems. However, the main focus of your talk should be on the representation theoretic consequences.

9. **The Lie Theorems I: Full faithfulness (I Nonis)** Dec 14, 2021
The goal of this talk and the one after it is to prove that taking associated Lie algebras gives rise to an equivalence of categories between simply connected Lie groups and finite-dimensional real Lie algebras. State this result and then prove the full faithfulness part, in particular introducing the Baker-Campbell-Hausdorff formula. You can follow e.g. [Tit83, III.4 and II.4.2–3] and [HN12, 9.5.2 and 3.4].

Note. The sources mentioned above work with analytic Lie groups while we work with smooth ones. You should briefly explain (but definitely not prove) why this distinction is irrelevant [DK00, Theorem 1.6.3, Proposition 1.6.4, and Corollary 1.10.9].

10. **The Lie Theorems II: Essential surjectivity (B Zondervan)** Dec 21, 2021
The goal of this talk is to prove that every finite-dimensional real Lie algebra comes from a simply connected Lie group (also called the Cartan-Lie Theorem). The main focus of your talk should be the solvable case following [Tit83, IV.1], in particular proving Lie's Theorem on complex representations of solvable Lie groups.
Afterwards, define semisimple Lie algebras and explain (rather briefly) how to build a simply connected Lie group from a semisimple Lie algebra using the Integral Subgroup Theorem [HN12, Theorem 9.4.8] and that the adjoint representation of a semisimple Lie algebra is faithful. Finally, state the Levi decomposition expressing general Lie algebras as a semi-direct product of a solvable and a semisimple one and use this to prove the theorem in full generality [Kna02, Appendix B].

11. **Classification of semisimple Lie groups and algebras I (M Kleinau)** Jan 11, 2022
Introduce/recall simple and semisimple Lie algebras and explain how these notions relate to each other [Kna02, I.2]; moreover, introduce semisimple Lie groups as on [Kna02, p. 61]. Prove Cartan's characterization of semisimple Lie algebras in terms of the Killing form following [Kna02, I.7] or [Hum80, II.4.3–5.1].
Afterwards, state the classification result of complex simple Lie algebras [Hum80, I.1.2, III.11.4, and V.19.2] (leaving the exceptional Lie algebras implicit), and indicate how this yields the classification of simply connected compact semisimple Lie groups [Tit83, IV.6.4]. If time permits, sketch the classification over the real numbers [Kna02, VI.8-10] or [Tit83, IV.6.1], hence of simply connected semisimple Lie groups, and very briefly indicate how this can be used to classify all connected semisimple (not necessarily simply connected) Lie groups [HN12, 9.5.1], also cf. the introduction of [Bt85, V.8].

12. **Classification of semisimple Lie groups and algebras II (T Zhang)** Jan 18, 2022
Introduce the abstract Jordan decomposition in a semisimple Lie algebra [Hum80, II.5.4] and compare it to the classical one for subalgebras of $\mathfrak{gl}_n(\mathbb{C})$ [Hum80, II.6.4]. Afterwards, study maximal toral subalgebras, introduce 'the' root system of a semisimple Lie algebra, and establish its basic properties following [Hum80, II.8]. Alternatively, you can follow [Kna02, II.1–4] for this talk (which uses an a priori different, but equivalent approach).

13. **Classification of semisimple Lie groups and algebras III (N Wolf)** Jan 25, 2022
Introduce abstract root systems as well as their Cartan matrices and Dynkyn diagrams following e.g. [Kna02, II.5] or [Hum80, III.9–11.3]. The main focus of your talk should then be a detailed account of the classification of Dynkyn diagrams as presented in [Kna02, II.7] or [Hum80, III.11.4]. Explain (without proof!) how this leads to the classification of semisimple complex Lie algebras [Hum80, V.18.4].
If you have time left, indicate why the classical Lie algebras of types A_n to D_n are indeed semisimple, and maybe explicitly calculate the root systems and Dynkyn diagrams for one of these families [Hum80, V.19.1–2].

Note. A quick tour through the main topics of the final three talks is given in [Bt85, V.5].

References

[Bt85] Theodor Bröcker and Tammo tom Dieck, *Representations of Compact Lie Groups*, Grad. Texts Math., vol. 98, Springer, New York, NY, 1985.

[DK00] J. J. Duistermaat and J. A. C. Kolk, *Lie Groups*, Universitext, Berlin: Springer, 2000.

[Hal74] Paul R. Halmos, *Measure Theory*, 2nd ed., Grad. Texts Math., vol. 18, Springer, New York, NY, 1974.

[HN12] Joachim Hilgert and Karl-Hermann Neeb, *Structure and Geometry of Lie Groups*, Springer Monogr. Math., Berlin: Springer, 2012.

[Hum80] James E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, 3rd revised ed., Grad. Texts Math., vol. 9, Springer, New York, NY, 1980.

[Kna02] Anthony W. Knapp, *Lie Groups Beyond an Introduction*, 2nd ed., Prog. Math., vol. 140, Boston, MA: Birkhäuser, 2002.

[Lan02] Serge Lang, *Algebra*, 3rd revised ed., Grad. Texts Math., vol. 211, New York, NY: Springer, 2002.

[Tit83] Jacques Tits, *Liesche Gruppen und Algebren. Unter Mitarbeit von M. Kraemer und H. Scheerer*, Hochschultext, Berlin-Heidelberg-New York-Tokyo: Springer-Verlag, 1983 (German).

[tom09] Tammo tom Dieck, *Representation Theory*, Lecture notes, available at `https://www.uni-math.gwdg.de/tammo/rep.pdf`, 2009.

[vQ01] Boto von Querenburg, *Mengentheoretische Topologie*, 3rd revised ed., Berlin: Springer, 2001 (German).

In mathematics and theoretical physics, a representation of a Lie group is a linear action of a Lie group on a vector space. Equivalently, a representation is a smooth homomorphism of the group into the group of invertible operators on the vector space. Representations play an important role in the study of continuous symmetry. A great deal is known about such representations, a basic tool in their study being the use of the corresponding 'infinitesimal' representations of Lie algebras. *From algebra representation to group representation. Representations of su(2). Canonical form for the adjoint representation. Irreducible representations of su(2). Compact simple Lie algebras and Cartan subalgebras. Cartan subalgebra. Weights and roots.Â Matrix Lie groups lead to real Lie algebras, as we have seen, but Lie algebras can be dened and studied in their own right. (Lie algebras do not uniquely determine corresponding Lie groups). Notation. This article gives a table of some common Lie groups and their associated Lie algebras. The following are noted: the topological properties of the group (dimension; connectedness; compactness; the nature of the fundamental group; and whether or not they are simply connected) as well as on their algebraic properties (abelian; simple; semisimple). For more examples of Lie groups and other related topics see the list of simple Lie groups; the Bianchi classification of groups of up to three dimensions... Lie algebras and their representation. We want to introduce Lie algebras as tangent spaces at the identity for Lie groups. As for any differential manifold, the tangent space at a point.Â an open set in the Lie group. The exponential map plays a key role in connecting Lie algebras representation to Lie groups representation. So we want to study Lie algebras reps, first, and then furnish a way to refer them to corresponding Lie groups. Definition. Let.