MALLIAVIN CALCULUS - DESCRIPTION OF THE LECTURE COURSE

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" Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning." (Albert Einstein, Autobiographical Notes 1949)

1. Introduction

Malliavin Calculus has been established in the second half of the twentieth century in order to provide a probabilistic approach to several problems from finite and infinite dimensional analysis. It is difficult to give an overview of the main applications of the theory, however, I want to present several characteristic examples, which can be treated by the methods of the lecture course. Methods of Malliavin Calculus are of great interest in recent research, which will be demonstrated, too.

In the lecture course all concepts will be developed from a basic knowledge in functional analysis and probability theory. Malliavin Calculus is an amazing subject, since the links between probability theory, geometry and (functional) analysis are illuminated. It is deep mathematics of the twentieth century with all of its modern methodology. Even though there are relations to several mathematical areas (from numerical mathematics to differential geometry), I want to point out mainly applications in economics and financial mathematics, but I am open for all suggestions of examples to be treated.

Several concepts of Malliavin Calculus seem at first sight abstract and strange, since "covariance" (wrinting stochastic differential equations in geometric terms) starts to play a role - here the difficulties, which Albert Einstein describes, appear again.

1.1. **Hedging Strategies in complete Markets.** Given the (simple) market model $(B_t, S_t) = (\exp(rt), \exp(\mu t + \sigma W_t - \frac{1}{2}\sigma^2 t))$ for a financial market with riskless bank-account B and risky asset S (modeled by a geometric Brownian motion), we are interested in self-financing hedging strategies for contingent claims X, i.e. adapted stochastic processes (ϕ_t, ψ_t) such that

$$\begin{aligned} V_t &= \phi_t B_t + \psi_t S_t \\ dV_t &= \phi_t dB_t + \psi_t dS_t \\ V_T &= X \end{aligned}$$

Calculating such strategies amounts to calculating the uniquely given integrand ψ_t of a stochastic integral

$$X = E(X) + \int_0^T \psi_t dS_t$$

which can be done by the Clark-Hausmann-Ocone-Formula of Malliavin Calculus explicitly. This method and its generalizations provide in several cases a good alternative to "classical" hedging methods, see also the famous article [KO91].

1.2. **Hörmander's "Sum of the squares"-theorem.** Given a second order elliptic partial differential operator $\mathcal{L}$ on $\mathbb{R}^n$

$$\mathcal{L} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$$

we are interested in solutions of the equation

$$\frac{\partial}{\partial t} f(t,x) = \mathcal{L} f(t,x)$$
$$f(0,x) = f_0(x)$$

for $t \geq 0$ and $x \in \mathbb{R}^n$. We will provide an appropriate functional analytic formulation of the problem and the probabilistic view point. Then we address the questions, whether the solution can be expressed as an integral operator

$$f(t,x) = \int_{\mathbb{R}^n} k(x,y,t) f_0(y) dy$$

and whether the kernel k has good regularity properties (e.g. smoothness)? The question is obviously highly complicated and depends roughly on the coefficients of the operator $\mathcal{L}$ - there is a hidden geometric question behind behind (distributions and Frobenius theory are applied). This application made Malliavin Calculus famous. See for example [Nua95], chapter 2.

The results concerning the existence of a density k can be applied in the context of portfolio optimization (for piecewise concave criteria functions). In this case absolute continuity with respect to Lebesgue measure of the law of a process $(Z_t^0)_{0 \leq t \leq T}$ at time T is desireable and characterized by the above methods. These are recent results by Laurence Carassus and Huyen Pham.

1.3. **Cubature on Wiener Space.** This is an interesting example for the methods from the lecture course (algebraic properties of the chaos decomposition), which recently lead to some numerical methods on Wiener space. The classical approximations of definite integrals

$$\int_a^b f(x) dx \approx \sum f(\alpha_i) \lambda_i$$

with $a \leq \alpha_i \leq b$ and weigths λ_i can be extended to Wiener space, which is done by Terry Lyons and Nicolas Victoir. It seems to provide some satisfying numerical procedures for the evaluation of expectations of iterated Stratonovich integrals. This can be applied to calculate expectations on Wiener space. Polynomials in the classical setting are replaced by multiple Wiener-Stratonovich-integrals

1.4. **Geometry of financial markets.** In the world of Stochastic Differential Geometry (see for example [Mal97]) classical geometric notions as curvature can be associated to financial markets. These interpretations lead to interesting estimates of hedging strategies in financial markets. This is recent work by A.B. Cruzeiro and P. Malliavin.

2. Contents and Literature

I start with minimal prerequisities as basic functional analysis and basic probability theory, hence I will introduce during the lecture course Brownian motion, Ito's integral, stochastic differential equations, strongly continuous semigroups, associated Sobolev spaces and basic results on partial differential equations. Most of the results will be proved in the course.

I shall follow the textbook of Nualart [Nua95], chapter 1 and chapter 2, in my presentation of the material, even though I shall apply slightly different notations. Since I want to continue either in a forthcoming lecture course or at the end of this lecture with white noise calculus in the spirit of the book of Holden, Oksendal, Uboe and Zhang [HØUZ96], it is necessary to use a more universal notation. Both books can be warmly recommended. For anybody interested in a more abstract introduction to Malliavin Calculus, I recommend the lecture notes of Peter Imkeller on Malliavin Calculus (Der stochastische Variationskalkül und Anwendungen in der multiplikativen Ergodentheorie), which can be found on his homepage ("http://wws.mathematik.hu-berlin.de/~imkeller/lectures.html"), and the (difficult, but ingenious) books of Paul Malliavin (for example [Mal95] and [Mal97]). In all of the recommended books one can find extensive references, which can serve for further studies.

The lecture course will be roughly structured in the following way:

1. Hilbert spaces, Normal random variables, Gaussian spaces, Isonormal Gaussian processes, Abstract chaos decomposition.
2. Brownian motion, Ito integration, multiple Wiener-Ito-integrals.
3. Wiener Chaos decomposition.
4. Girsanov's theorem and its consequences (other approaches to Malliavin Calculus).
5. Malliavin derivative, closed operators on Banach spaces, operator norms.
6. Skohorod integrals, adjoint operators, local properties, the Ito integral as particular case of the Skohorod integral.
7. The Clark-Hausmann-Ocone-Formula, Karatzas-Ocone-Hedging.
8. C_0-Semigroups on Banach spaces, Ornstein-Uhlenbeck-Semigroup, Hypercontractivity.
9. Sobolev spaces and equivalence of norms.
10. Stochastic Differential equations, Markov properties.
11. Hypoellipticity and Hörmander's theorem.
12. Existence and smoothness of densities.
13. White Noise Calculus.
14. Stochastic Differential Geometry.

In the lecture course several carefully examples chosen will be regularly treated, such that everybody can get experienced with the problems of Malliavin Calculus. Many examples are chosen, besides from the already cited literature, are taken from Bernt Oksendal's "An introduction to Malliavin Calculus with applications to economics" (which can be found under "http://www.nhh.no/for/dp/1996/").

If time permits, I will provide some lecture notes.

References

[HØUZ96] H. Holden, B. Øksendal, J. Ubøe, and T. Zhang, *Stochastic Partial Differential Equations*, Probability and its Applications, Birkhäuser, 1996.

[KO91] Ioannis Karatzas and Daniel Ocone, *A generalized Clark representation formula with applications to optimal portfolios*, Stochastics and Stochastics Report **34** (1991), 187–220.

[Mal95] P. Malliavin, *Integration and Probability*, Graduate Texts in Mathematics, Springer-Verlag, 1995.

[Mal97] P. Mallavin, *Stochastic Calculus*, Grundlagen der Mathematischen Wissenschaften, vol. 313, Springer-Verlag, 1997.

[Nua95] D. Nualart, *The Malliavin Calculus and Related Topics*, Probability and its Applications, Springer-Verlag, 1995.

Coloured Noise MALLIAVIN REGULARITY OF SOLUTIONS OF SPDEs ANALYSIS OF THE MALLIAVIN MATRIC OF SOLUTIONS OF SPDEs One Dimensional Case Examples Multidimensional Case DEFINITION OF SPACES USED THROUGHOUT THE COURSE. Discover the world's research. 20+ million members.Â The Malliavin derivative D, the Skorokhod integral Î´ and the Ornstein-Uhlenbeck operator R are three operators that play a crucial role in the stochastic calculus of variations, an infinite-dimensional differential calculus on white noise spaces [2,7,36,42,43,48]. These operators correspond respectively to the annihilation, the creation and the number operator in quantum operator theory. ... The Malliavin calculus generalises in a natural way to Hilbert space-valued random variables. We refer to [6] for a recent account of this innite dimensional setting with applications to stochastic (partial) dierential equations. In recent years many Hilbert space results in stochastic (and harmonic) analysis have been transferred to a Banach space setting [11, 13]. Of particular relevance for this work is the theory of stochastic integration in Banach spaces developed by van Neerven, Veraar and Weis [24, 26].Â It has been proved by Pisier [31] that the fundamental Meyer inequalities remain valid if the Banach space is a UMD space, provided that the norm of the derivative is taken in the appropriate space. An Introduction to Malliavin Calculus. Courant Institute of Mathematical Sciences New York University. Peter K. Friz. August 3, 2002. These notes available on www.math.nyu.edu/phd students/frizpete Please send corrections to Peter.Friz@cims.nyu.edu. These lecture-notes are based on a couple of seminar-talks I gave at Courant in Spring 2001. I am extremely grateful to Prof. S.R.S. Varadhan for supporting and stimulating my interest in Malliavin Calculus.Â Remarks: - A slightly dierent description of of the Wiener-Chaos, Cn = closure of span{Jn(hâŠ—n) : h H = 1} = closure of span{Hn(W (h)) : h H = 1}. (1.16). The Malliavin calculus was developed to provide a probabilistic proof of Hormander's hypoellipticity theorem. The theory has expanded to encompass other significant applications. The main application of the Malliavin calculus is to establish the regularity of the probability distribution of functionals of an underlying Gaussian process. In this way, one can prove the existence and smoothness of the density for solutions of various stochastic differential equations. More recently, applications of the Malliavin calculus in areas such as stochastic calculus for fractional Brownian motion, central limit theorems for multiple stochastic integrals, and mathematical finance have emerged. The first part of the book covers the basic results of the Malliavin calculus. Malliavin calculus is a stochastic calculus of variations on the Wiener space. Its foundations were set in the 1970â€™s, mainly in the seminal work [33], in order to study the existence and smoothness of density for the probability laws of random vectors. For diusion processes this problem can be approached by applying HÂ¨ormanderâ€™s theorem on hy-poelliptic dierential operators in square form to Kolmogorovâ€™s equation (see ref. [18]).Â At a more applied level, Malliavin calculus is used in probabilistic numerical meth-ods in nancial mathematics.Â [68]) yields â€” under stronger assumptions â€” a description of the densities and their derivatives. We present here a review of these results, putting more emphasis on the second approach. Let us rst introduce some notation.