APPROXIMATE GROUPS AND ERDŐS GEOMETRY

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For the general area of arithmetic combinatorics a good introductory book is [TV10] by Tao and Vu. For the connections with Model Theory, see Hrushovski's original paper [Hru12] and his survey [Hru13]. Tao's blog also contains precious and accessible information on most of the topics discussed in this mini-course.

Surveys and books : [Hru13], [Dvi10], [TV10], [Bre15], [Bre14], [Bre16], [BGT13].

Selected original papers: [Tao08], [Tao15], [Hru12], [BGT11], [BGT12], [Tao15], [ES12]

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Geometric questions which involve Euclidean distances often lead to polynomial relations of type F(x, y, z)=0 for some F âˆˆ â„[x, y, z]. Several problems of Combinatorial Geometry can be reduced to studying such polynomials which have many zeroes on nÃ—nÃ—n Cartesian products. The special case when the relation F = 0 can be re-written as z = f(x, y), for a polynomial or rational function f âˆˆ â„(x, y), was considered in [8]. Our main goal is to extend the results found there to full generality (and also to show some geometric applications, e.g. one on â€œcircle gridsâ€). In the Guth/Katz solution to the Erdos Distance problem on N, we have that the minimum distances is given by considering an approximate grid. Let's have $N=n^2$, so the grid is exactly the $n \times n$ grid. I am a little confused why this is the minimum? Doesn't this have at least $\frac{n(n+1)}{2}-1$ distinct distances? That is, the corner point has distinct distances to all points on the diagonal and all points above the diagonal (by symmetry all below have the same distance). I presume this calculation is wrong, because otherwise we clearly get the better solution of just taking regular polygons which have ~$\frac n 2$ distances. Edit: Internet sources suggest this is $\frac{N}{\sqrt{\log N}}$. 420 b . bollobÃ¡s and p. erdÅ‘s. Let d = d(n) be the positive real number for which. It is easily checked that. n =cd) i . (d)p, d(n) = 2log, n-2log b logb n+2log b (1e) + 1+0(1). b.Â If e is a line of this projective geometry, let Ge be the subgraph of G with point set V = U Ci and with all. C,ee. those edges of G that join points belonging to different classes . It is clear that almost. every r-tuple of V is such that no two points belong to the same class, since. Furthermore, \ g r 1 / mr- (, VrI. ,) . r < d(IVJ)-2 . Consequently inequality (7) implies that the probability of G e not containing a K, is less than n-1 .